

INVERTIBLE TOEPLITZ PRODUCTS, WEIGHTED NORM INEQUALITIES, AND A_p WEIGHTS

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ABSTRACT. In this paper, we characterize invertible Toeplitz products on a number of Banach spaces of analytic functions, including weighted Bergman space $L_a^p(\mathbb{B}_n, dv_\gamma)$, the Hardy space $H^p(\partial\mathbb{D})$, and the standard weighted Fock space F_α^p for $p > 1$. The common tool in the proofs of our characterizations will be the theory of weighted norm inequalities and A_p type weights. Furthermore, we prove weighted norm inequalities for the Fock projection, and compare the various A_p type conditions that arise in our results. Finally, we extend the “reverse Hölder inequality” of Zheng and Stroethoff [20, 21] for $p = 2$ to the general case of $p > 1$.

1. INTRODUCTION

Let \mathbb{B}_n denote the unit ball in \mathbb{C}^n and let dv denote the usual normalized volume measure on \mathbb{B}_n . For $\gamma > -1$, let $dv_\gamma(z) = c_\gamma(1 - |z|^2)^\gamma dv(z)$ where c_γ is a normalizing constant. For $1 \leq p < \infty$, the Bergman space $L_a^p(\mathbb{B}_n, dv_\gamma)$ is the Banach space of analytic functions on \mathbb{B}_n that belong to $L^p(\mathbb{B}_n, dv_\gamma)$.

As a (formal) limiting case $\gamma \rightarrow -1^+$ of the spaces $L_a^p(\mathbb{B}_n, dv_\gamma)$, one obtains the Hardy space $H^p(\partial\mathbb{B}_n)$, which is the closure in $L^p(\partial\mathbb{B}_n, d\sigma)$ of analytic polynomials on $\partial\mathbb{B}_n$ where $d\sigma$ is the standard surface measure on $\partial\mathbb{B}_n$ (more precisely, $dv_\gamma \xrightarrow{\text{wk}^*} d\sigma$ on $C(\overline{\mathbb{B}_n})$ as $\gamma \rightarrow -1^+$.) As another (formal) limiting case where $\gamma \rightarrow +\infty$, one obtains the Fock space F_α^p of all entire functions f where $f(\cdot)e^{-\frac{\alpha}{2}|\cdot|^2}$ is in $L^p(\mathbb{C}^n, (p\alpha/2\pi)^n dv)$ for $\alpha > 0$ (and where F_α^p is equipped with its canonical Banach space norm.)

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It is well known [26] that the orthogonal projection P_γ from $L^2(\mathbb{B}_n, dv_\gamma)$ onto $L_a^2(\mathbb{B}_n, dv_\gamma)$ is given by

$$P_\gamma f(z) = \int_{\mathbb{B}_n} K_\gamma(z, u) f(u) dv_\gamma(u)$$

where $K_\gamma(z, u)$ is the Bergman kernel $K_\gamma(z, u) = (1 - z \cdot u)^{-(n+1+\gamma)}$. Let $p > 1$ and let q be the conjugate exponent of p . If $g \in L^q(\mathbb{B}_n, dv_\gamma)$, then we can define the Toeplitz operator T_g with symbol g on $L_a^p(\mathbb{B}_n, dv_\gamma)$ by the formula $T_g = P_\gamma M_g$ (with M_g being “multiplication by g ”). Similarly, if $g \in L^q(\partial\mathbb{B}_n)$, then the Toeplitz operator T_g with symbol $g \in L^p(\partial\mathbb{B}_n)$ is defined on $H^p(\partial\mathbb{B}_n)$ by $T_g = P^+ M_g$ where P^+ is the Hardy projection. Note that while $T_g = P_\gamma M_g$ obviously depends on γ , for the sake of notational ease we will still refer to this Toeplitz operator on $L_a^p(\mathbb{B}_n, dv_\gamma)$ by T_g . The same will be true when we define Toeplitz operators on Fock spaces F_α^p in Section 3.

Toeplitz operators T_g on both the Hardy space and the Bergman space have been extensively studied in the literature when $p = 2$. (see [25] and the references therein.) However, it is well known [26] that both the Bergman projection P_γ and the Hardy projection P^+ are bounded on $L^p(\mathbb{B}_n, dv_\gamma)$ and $L^p(\partial\mathbb{B}_n, d\sigma)$, respectively, whenever $p > 1$. Thus, many of the results regarding Toeplitz operators for $p = 2$ can be appropriately generalized to the $p > 1$ case.

In [20, 21], the invertibility of the product of Toeplitz operators $T_f T_{\bar{g}}$ for analytic f and g was characterized for the Bergman space $L_a^2(\mathbb{B}_n, dv_\gamma)$ and the Hardy space $H^2(\partial\mathbb{B}_n)$ when $n = 1$. In particular, they proved the following result (where dA_γ is the weighted area measure on the unit disk \mathbb{D}):

Theorem 1.1. *For functions $f, g \in H^2(\partial\mathbb{D})$, the Toeplitz product $T_f T_{\bar{g}}$ is bounded and invertible on $H^2(\partial\mathbb{D})$ if and only if*

$$\inf_{u \in \mathbb{D}} |f(u)| |g(u)| > 0$$

and

$$\sup_{u \in \mathbb{D}} \widehat{|f|^2}(u) \widehat{|g|^2}(u) < \infty.$$

Moreover, for $f, g \in L_a^2(\mathbb{D}, dA_\gamma)$, $T_f T_{\bar{g}}$ is bounded and invertible on $L_a^2(\mathbb{D}, dA_\gamma)$ if and only if

$$\inf_{u \in \mathbb{D}} |f(u)| |g(u)| > 0 \tag{1.1}$$

and

$$\sup_{u \in \mathbb{D}} B_\gamma(|f|^2)(u) B_\gamma(|g|^2)(u) < \infty. \quad (1.2)$$

Here, \widehat{f} is the Poisson extension of a function f on $\partial\mathbb{D}$ and $B_\gamma f$ is the Berezin transform of a function f on \mathbb{D} given by

$$B_\gamma(f)(z) = \int_{\mathbb{D}} f(u) |k_z^\gamma(u)|^2 dA_\gamma(u)$$

where k_z^γ is the normalized Bergman kernel $k_z^\gamma(u) = K_\gamma(u, z)/K_\gamma(z, z)$ for $L_a^p(\mathbb{D}, dA_\gamma)$. For the sake of notational ease, we will drop the γ in the notation for k_z^γ in the rest of the paper.

The main step in proving Theorem 1.1 (in both the Bergman and Hardy space settings) is showing that the hypotheses in Theorem 1.1 are enough to guarantee the boundedness of $T_f T_{\overline{g}}$. Once this is done, then an easy argument from [20, 21] completes the proof.

To prove the boundedness of $T_f T_{\overline{g}}$, the authors first proved in [21] that for $f, g \in L_a^2(\mathbb{D}, dA_\gamma)$, we have that $T_f T_{\overline{g}}$ is bounded on $L_a^2(\mathbb{D}, dA_\gamma)$ if there exists $\epsilon > 0$ such that

$$\sup_{u \in \mathbb{D}} B_\gamma(|f|^{2+\epsilon})(u) B_\gamma(|g|^{2+\epsilon})(u) < \infty.$$

The authors then proved that $T_f T_{\overline{f^{-1}}}$ is bounded by showing that there exists some $\epsilon > 0$ where

$$\sup_{u \in \mathbb{D}} B_\gamma(|f|^{2+\epsilon})(u) B_\gamma(|f|^{-(2+\epsilon)})(u) < \infty$$

whenever (1.2) holds for $g = f^{-1}$ (which is true if (1.1) and (1.2) hold.) The boundedness of $T_f T_{\overline{g}}$ then follows easily from this fact and conditions (1.1) and (1.2). For the boundedness of the Toeplitz product $T_f T_{\overline{g}}$ on the Hardy space, the authors use the same argument and Theorem 8 from [24].

It was remarked in [6], however, that the boundedness of $T_f T_{\overline{g}}$ on either the Hardy space $H^2(\partial\mathbb{D})$ or the Bergman space $L_a^2(\mathbb{D}, dv_\gamma)$ for analytic f and g is equivalent to the boundedness of the Hardy projection P^+ (respectively, the Bergman projection P_γ) from the weighted space $L^2(\partial\mathbb{D}, |g|^{-2} d\sigma)$ to the weighted space $L^2(\partial\mathbb{D}, |f|^2 d\sigma)$ (where the obvious changes are made for the Bergman space.)

More generally, the boundedness of the Hardy projection P^+ on $L^p(\partial\mathbb{B}_n, d\sigma)$ tells us that for any symbols f and g (not necessarily analytic), $T_f T_{\overline{g}}$ is bounded on the Hardy space $H^p(\partial\mathbb{B}_n)$ (in fact, bounded on $L^p(\partial\mathbb{B}_n, d\sigma)$)

if P^+ is bounded from $L^p(\partial\mathbb{B}_n, |g|^{-p}d\sigma)$ to $L^p(\partial\mathbb{B}_n, |f|^p d\sigma)$. Moreover, a similar result holds for the boundedness of $T_f T_{\bar{g}}$ on $L_a^p(\mathbb{B}_n, dv_\gamma)$.

Unfortunately, the “two-weight” problem of characterizing the weights w and v on $\partial\mathbb{B}_n$ where P^+ is bounded from $L^p(\partial\mathbb{B}_n, w d\sigma)$ to $L^p(\partial\mathbb{B}_n, v d\sigma)$ is very difficult and poorly understood even for $n = 1$ (the “two-weight” problem for P^+ when $n = 1$ can be found in [7, 8], but their condition is extremely difficult to work with and is thus far from optimal.) Furthermore, a similar statement can be said about the corresponding problem for the Bergman projection on \mathbb{B}_n .

If $w = v$, however, it is well known that P^+ is bounded on $L^p(\partial\mathbb{D}, w d\sigma)$ if and only if w satisfies the Muckenhoupt A_p condition. Similarly, it is well known that P_γ is bounded on $L^p(\mathbb{B}_n, w dv_\gamma)$ if and only if w satisfies the Békollé - Bonami condition $B_{p,\gamma}$ (both of these conditions will be defined in the next section.)

In the next section, we will combine weighted norm inequalities for the Hardy and Bergman projections with ideas from [20, 21] to characterize bounded and invertible $T_f T_{\bar{g}}$ on both the Hardy space $H^p(\partial\mathbb{D})$ and the Bergman space $L_a^p(\mathbb{B}_n, dv_\gamma)$ when f and g are analytic. It should be noted that not only is this approach much simpler than the one taken in [20, 21], but it also provides us with a blueprint for characterizing bounded and invertible Toeplitz products on the Fock space F_α^p for $p > 1$.

In particular, in Section 3, we will characterize weights w on \mathbb{C}^n where the Fock projection (which will be defined in Section 3) is bounded on the weighted space $\mathcal{L}_\alpha^p(w)$. Here, $\mathcal{L}_\alpha^p(w)$ is the Banach space (equipped with its canonical Banach space norm) of all f where $f(\cdot)e^{-\frac{\alpha}{2}|\cdot|^2} \in L^p(\mathbb{C}^n, w dv)$ for $\alpha > 0$. Also we will use the general arguments from Section 2, along with our weighted norm inequalities for the Fock projection, to characterize bounded and invertible Toeplitz products on F_α^p . As a trivial consequence of these results, we will show that “Sarason’s conjecture” on the product of Toeplitz operators is trivially true for the Fock space F_α^p , which is in stark contrast to the Hardy space where it is known that Sarason’s conjecture is false (see [6] for detailed information about Sarason’s conjecture, and see [16] for a counterexample in the Hardy space case).

In Section 4, we will discuss in some detail the various classes of weights used in Sections 2 and 3, and also discuss connections between these classes.

It should be noted that although the theory of weighted norm inequalities simplifies the arguments in [20, 21], the techniques developed in these two papers (in particular, their “reverse Hölder inequality” and the Calderon-Zygmund decomposition adapted to the hyperbolic disk) are of independent interest themselves. Thus, in our last section (Section 5), we will present a proof of our characterization of invertible Toeplitz products on the Bergman space $L_a^p(\mathbb{D}, dA_\gamma)$ that extends these techniques to handle the general case $p > 1$, rather than just the $p = 2$ case. In particular, we will extend the “reverse Hölder inequality” of Zheng and Stroethoff [20, 21] for $p = 2$ to the general case of $p > 1$. It is hoped that the ideas in Section 5 will have applications to other Bergman space problems where Möbius invariance is unavailable, or where classical Calderon-Zygmund theory techniques are relevant.

Finally, throughout the paper we will let C denote a constant that may change from line to line (or even on the same line.)

2. INVERTIBLE TOEPLITZ PRODUCTS ON THE HARDY AND BERGMAN SPACES.

We will first discuss invertible Toeplitz products on the Bergman space $L_a^p(\mathbb{B}_n, dv_\gamma)$. The result we wish to prove is the following:

Theorem 2.1. *If $f \in L_a^p(\mathbb{B}_n, dv_\gamma)$ and $g \in L_a^q(\mathbb{B}_n, dv_\gamma)$, then the Toeplitz product $T_f T_{\bar{g}}$ is bounded and invertible on $L_a^p(\mathbb{B}_n, dv_\gamma)$ if and only if*

$$\inf_{u \in \mathbb{B}_n} |f(u)| |g(u)| > 0$$

and

$$\sup_{u \in \mathbb{B}_n} \{B_\gamma(|f k_u^{1-\frac{2}{p}}|^p)(u)\}^{\frac{1}{p}} \{B_\gamma(|g k_u^{1-\frac{2}{q}}|^q)(u)\}^{\frac{1}{q}} < \infty.$$

Before we prove this, we will need to discuss the Békollé - Bonami class $B_{p,\gamma}$. For $z, u \in \mathbb{B}_n$, let d be the pseudo-metric on \mathbb{B}_n given by $d(z, u) = ||z| - |u|| + |1 - \frac{z}{|z|} \cdot \frac{u}{|u|}|$ and let $D = D(z, R)$ denote a ball in \mathbb{B}_n with respect to this pseudo-metric. We say that a weight w on \mathbb{B}_n is in $B_{p,\gamma}$ if

$$\left(\frac{1}{v_\gamma(D)} \int_D w dv_\gamma \right) \left(\frac{1}{v_\gamma(D)} \int_D w^{-\frac{1}{p-1}} dv_\gamma \right)^{p-1} < C \quad (2.1)$$

where D is any such ball that intersects $\partial\mathbb{B}_n$ and C is independent of D .

The following theorem was proved in [3], which solves the “one-weight” problem for the Bergman projection P_γ :

Theorem 2.2. *The Bergman projection P_γ is bounded on the weighted space $L^p(\mathbb{B}_n, w dv_\gamma)$ if and only if $w \in B_{p,\gamma}$.*

We will also need the following result found in [15]

Theorem 2.3. *If $f \in L_a^p(\mathbb{B}_n, dv_\gamma)$ and $g \in L_a^q(\mathbb{B}_n, dv_\gamma)$, then*

$$\sup_{u \in \mathbb{B}_n} \{B_\gamma(|fk_u^{1-\frac{2}{p}}|^p)(u)\}^{\frac{1}{p}} \{B_\gamma(|gk_u^{1-\frac{2}{q}}|^q)(u)\}^{\frac{1}{q}} < \infty$$

whenever $T_f T_{\bar{g}}$ is bounded on $L_a^p(\mathbb{B}_n, dv_\gamma)$.

With the aid of Theorem 2.2 and 2.3, we can now prove Theorem 2.1:

Proof of Theorem 2.1: First we will prove necessity. The proof of this direction is similar to the corresponding result in [20, 21], though we include it for the sake of completion. Assume that $T_f T_{\bar{g}}$ is bounded and invertible on $L_a^p(\mathbb{B}_n, dv_\gamma)$, so that $(T_f T_{\bar{g}})^* = T_g T_{\bar{f}}$ is bounded and invertible on $L_a^q(\mathbb{B}_n, dv_\gamma) = (L_a^p(\mathbb{B}_n, dv_\gamma))^*$. Let $C_1 = \|(T_f T_{\bar{g}})^{-1}\|_p$ and $C_2 = \|(T_g T_{\bar{f}})^{-1}\|_q$. First note that $T_f T_{\bar{g}} k_u = \overline{g(u)} f k_u$, so that

$$\begin{aligned} \|k_u\|_p &\leq C_1 \|T_f T_{\bar{g}} k_u\|_p \\ &= C_1 |g(u)| \{B_\gamma(|fk_u^{1-2/p}|^p)(u)\}^{\frac{1}{p}} \end{aligned}$$

Similarly, since $(T_f T_{\bar{g}})^* = T_g T_{\bar{f}}$ is bounded and invertible on $L_a^q(\mathbb{B}_n, dv_\gamma) = (L_a^p(\mathbb{B}_n, dv_\gamma))^*$, we have that

$$\|k_u\|_q \leq C_2 |f(u)| \{B_\gamma(|gk_u^{1-2/q}|^q)(u)\}^{\frac{1}{q}}.$$

By Theorem 2.2, we have that

$$\{B_\gamma(|fk_u^{1-2/p}|^p)(u)\}^{\frac{1}{p}} \{B_\gamma(|gk_u^{1-2/q}|^q)(u)\}^{\frac{1}{q}} \leq M \quad (2.2)$$

for some $M > 0$ independent of u . Moreover, an application of Hölder's inequality gives us that $\|k_u\|_p \|k_u\|_q \geq 1$ for any $u \in \mathbb{B}$, which tells us that

$$C_1 C_2 M |f(u)| |g(u)| \geq \|k_u\|_p \|k_u\|_q \geq 1$$

which means that $\inf_{u \in \mathbb{B}_n} |f(u)| |g(u)| > 0$

Now we will prove sufficiency. Let M be the constant in (2.2) and let

$$\eta = \inf_{u \in \mathbb{B}_n} |f(u)| |g(u)|.$$

Let φ_u be the Möbius transformation that interchanges 0 and u . By Hölder's inequality, we have that

$$\begin{aligned} |f(u)| &= (1 - |u|^2)^{\left(\frac{2+\gamma}{2}\right)\left(1-\frac{2}{p}\right)} |f \circ \varphi_u(0)| |k_u \circ \varphi_u(0)|^{1-\frac{2}{p}} \\ &\leq (1 - |u|^2)^{\left(\frac{2+\gamma}{2}\right)\left(1-\frac{2}{p}\right)} \{B_\gamma(|fk_u^{1-2/p}|^p)(u)\}^{\frac{1}{p}}. \end{aligned} \quad (2.3)$$

and similarly

$$|g(u)| \leq (1 - |u|^2)^{\left(\frac{2+\gamma}{2}\right)\left(1-\frac{2}{q}\right)} \{B_\gamma(|gk_u^{1-2/q}|^q)(u)\}^{\frac{1}{q}}$$

which means that

$$\sup_{u \in \mathbb{B}_n} |f(u)| |g(u)| \leq M$$

Also, since $|g(u)|^q \geq \eta^q |f^{-1}(u)|^q$ we have that

$$\{B_\gamma(|f^{-1}k_u^{1-2/q}|^q)(u)\}^{\frac{1}{q}} \leq \eta^{-q} \{B_\gamma(|gk_u^{1-2/q}|^q)(u)\}^{\frac{1}{q}}$$

which means that

$$\sup_{u \in \mathbb{B}_n} \{B_\gamma(|fk_u^{1-\frac{2}{p}}|^p)(u)\}^{\frac{1}{p}} \{B_\gamma(|f^{-1}k_u^{1-\frac{2}{q}}|^q)(u)\}^{\frac{1}{q}} < \infty. \quad (2.4)$$

If $w = |f|^p$, then it is easy to see that (2.4) and Lemma 2 in [3] implies that $w \in B_{p,\gamma}$, so that $T_f T_{\bar{g}}$ is bounded on $L_a^p(\mathbb{B}_n, dv_\gamma)$. Also, since $\phi = (f\bar{g})^{-1}$ is bounded, T_ϕ is bounded on $L_a^p(\mathbb{B}_n, dv_\gamma)$. Moreover, it is easy to check that

$$T_f T_{\bar{g}} T_\phi = I = T_\phi T_f T_{\bar{g}}$$

which completes the proof. \square

We will now prove the Hardy space version of Theorem 2.1. First, recall that the Muckenhoupt class A_p is the collection of all weights w on $\partial\mathbb{D}$ where

$$\sup_{I \subseteq \partial\mathbb{D}} \left(\frac{1}{|I|} \int_I w d\theta \right) \left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} d\theta \right)^{p-1} < \infty \quad (2.5)$$

and where the supremum is taken over all arcs $I \subseteq \partial\mathbb{D}$. It is well known [4] that the Hardy projection P^+ is bounded on $L^p(\partial\mathbb{D}, w d\theta)$ if and only if $w \in A_p$. With this result, we will now prove the following:

Theorem 2.4. *If $f \in H^p(\partial\mathbb{D})$ and $g \in H^q(\partial\mathbb{D})$, then the Toeplitz product $T_f T_{\bar{g}}$ is bounded and invertible on $H^p(\partial\mathbb{D})$ if and only if*

$$\inf_{u \in \mathbb{D}} |f(u)| |g(u)| > 0 \quad (2.6)$$

and

$$\sup_{u \in \mathbb{D}} \widehat{\{|fk_u^{1-\frac{2}{p}}|^p(u)\}^{\frac{1}{p}}} \widehat{\{|gk_u^{1-\frac{2}{q}}|^q(u)\}^{\frac{1}{q}}} < \infty.$$

Proof. First we prove necessity, so assume that $T_f T_{\bar{g}}$ is bounded and invertible on $H^p(\partial\mathbb{D})$. By an argument that is almost identical to the argument (due to S. Treil) in [19], we have that

$$\sup_{u \in \mathbb{D}} \widehat{\{|f k_u^{1-\frac{2}{p}}|^p(u)\}^{\frac{1}{p}}} \widehat{\{|g k_u^{1-\frac{2}{q}}|^q(u)\}^{\frac{1}{q}}} < \infty. \quad (2.7)$$

Thus, by an argument that is similar to the proof of Theorem 2.1, we have that

$$\inf_{u \in \mathbb{D}} |f(u)| |g(u)| > 0.$$

Now we prove sufficiency. Fix $u \in \mathbb{D}$ and let $f_r(u) = f(ru)$ for some fixed $0 < r < 1$. If we replace f with f_r in (2.3) then an elementary “calculus” argument allows us to take $\gamma \rightarrow -1^+$ in (2.3) to get that

$$|f(ru)| \leq (1 - |u|^2)^{\frac{1}{2}(1-\frac{2}{p})} \widehat{\{|f_r k_u^{1-\frac{2}{p}}|^p(u)\}^{\frac{1}{p}}}.$$

Thus, since $f \in H^p(\partial\mathbb{D})$, we can let $r \rightarrow 1^-$ to get

$$|f(u)| \leq (1 - |u|^2)^{\frac{1}{2}(1-\frac{2}{p})} \widehat{\{|f k_u^{1-\frac{2}{p}}|^p(u)\}^{\frac{1}{p}}}.$$

Applying the same inequality to g and using the hypothesis of Theorem 2.4, we have that

$$\sup_{u \in \mathbb{D}} |f(u)| |g(u)| < \infty. \quad (2.8)$$

Combining (2.6), (2.7) and (2.8) as we did in the proof of Theorem 2.1, it is easy to see that $w = |f|^p$ is in the Muckenhoupt A_p class, which implies that $T_f T_{\bar{g}}$ is bounded. Finally, since $\phi = (f \bar{g})^{-1}$ is bounded, it is again easy to see that

$$T_f T_{\bar{g}} T_{\phi} = I = T_{\phi} T_f T_{\bar{g}}$$

which implies that $T_f T_{\bar{g}}$ is invertible. \square

Remark 2.5. *It is known [14] that the Hardy projection P^+ is bounded on $L^p(\partial\mathbb{B}_n, w d\sigma)$ if and only if w satisfies (2.5) (where the supremum is taken over all non-isotropic balls in $\partial\mathbb{B}_n$). Furthermore, except for the proof that the boundedness of $T_f T_{\bar{g}}$ implies (2.7), the entire proof of Theorem 2.4 carries over to the case $n > 1$. Thus, we will conjecture that Theorem 2.4 holds for the unit sphere $\partial\mathbb{B}_n$ when $n > 1$.*

3. INVERTIBLE TOEPLITZ PRODUCTS AND WEIGHTED NORM INEQUALITIES FOR THE FOCK PROJECTION.

For any $\alpha > 0$, let \mathcal{L}_α^p be the Banach space of all f where $f(\cdot)e^{-\frac{\alpha}{2}|\cdot|^2} \in L^p(\mathbb{C}^n, (p\alpha/2\pi)^n dv)$. It is well known (see [13]) that the orthogonal projection P_α from \mathcal{L}_α^2 onto the Fock space F_α^2 is given by

$$P_\alpha f(z) = \int_{\mathbb{C}^n} e^{\alpha z \cdot u} f(u) d\mu_\alpha(u)$$

where $d\mu_\alpha$ is the Gaussian measure

$$d\mu_\alpha(u) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|u|^2} dv(u).$$

In this section, we will first state and prove weighted norm inequalities for the Fock projection P_α , and then use these weighted norm inequalities to characterize bounded and invertible Toeplitz products $T_f T_{\bar{g}}$ on the Fock space F_α^p for $p > 1$ when f and g are entire.

Let $Q_r(z)$ be the cube in \mathbb{C}^n with center z and side length $r > 0$. Let $A_{p,r}$ denote the class of weights w on \mathbb{C}^n where

$$\sup_{z \in \mathbb{C}^n} \left(\frac{1}{v(Q_r(z))} \int_{Q_r(z)} w dv \right) \left(\frac{1}{v(Q_r(z))} \int_{Q_r(z)} w^{-\frac{1}{p-1}} dv \right)^{p-1} < C_r \quad (3.1)$$

for some $0 < C_r < \infty$.

Theorem 3.1. *The following are equivalent for any weight w on \mathbb{C}^n and any $\alpha > 0$:*

- (a) $w \in A_{p,r}$ for some $r > 0$.
- (b) H_α is bounded on $L^p(\mathbb{C}^n, w dv)$.
- (c) P_α is bounded on $\mathcal{L}_\alpha^p(w)$
- (d) $w \in A_{p,r}$ for all $r > 0$.

Here, H_α is the integral operator given by

$$H_\alpha f(z) = \int_{\mathbb{C}^n} e^{-\frac{\alpha}{2}|z-u|^2} f(u) dv(u).$$

We will need three simple lemmas to prove Theorem 3.1. It should be noted that the proofs of the first two lemmas use standard arguments from the classical theory of weighted norm inequalities. In what follows, we will let

$$w(S) := \int_S w dv$$

for any measurable $S \subseteq \mathbb{C}^n$.

Lemma 3.2. *Let $Q_r = Q_r(z)$ be any cube in \mathbb{C}^n of side length r , and let $3Q_r$ denote the cube with the same center but with side length $3r$. If $w \in A_{p,3r}$, then $w(3Q_r) \leq Cw(Q_r)$ for some constant $C > 0$ independent of Q_r (but obviously depending on r .)*

Proof. By Hölder's inequality and (3.1), there exists $C > 0$ such that

$$\begin{aligned} r^{2n} &= \int_{Q_r} w^{1/p} w^{-1/p} dv \\ &\leq (w(Q_r))^{\frac{1}{p}} \left(\int_{Q_r} w^{-\frac{1}{p-1}} dv \right)^{(p-1)/p} \\ &\leq \left(\frac{w(Q_r)}{w(3Q_r)} \right)^{\frac{1}{p}} \left(\int_{3Q_r} w dv \right)^{\frac{1}{p}} \left(\int_{3Q_r} w^{-\frac{1}{p-1}} dv \right)^{(p-1)/p} \\ &\leq C \left(\frac{w(Q_r)}{w(3Q_r)} \right)^{\frac{1}{p}} \end{aligned}$$

where C is independent of Q_r . \square

Lemma 3.3. *Let Q_r be any cube in \mathbb{C}^n of side length r and let f be any measurable function on \mathbb{C}^n . If $w \in A_{p,3r}$, then there exists $C > 0$ independent of Q_r and f where*

$$\left(\int_{Q_r} |f| dv \right)^p \leq C \frac{1}{w(Q_r)} \int_{Q_r} |f|^p w dv$$

Proof. The proof is similar to the proof of Lemma 3.2. In particular, since $A_{p,3r} \subseteq A_{p,r}$, there is some $C > 0$ independent of Q_r where

$$\begin{aligned} \left(\int_{Q_r} |f| dv \right)^p &\leq \left(\int_{Q_r} |f|^p w dv \right) \left(\int_{Q_r} w^{-\frac{1}{p-1}} dv \right)^{p-1} \\ &\leq C \frac{1}{w(Q_r)} \int_{Q_r} |f|^p w dv \end{aligned}$$

\square

For the next lemma we will need the notion of a discrete path from [12]. For each $r > 0$, let $r\mathbb{Z}^{2n}$ denote the set $\{(rk_1, \dots, rk_{2n}) \in \mathbb{R}^{2n} : k_i \in \mathbb{Z}\}$. Since \mathbb{R}^{2n} can canonically be identified with \mathbb{C}^n , we will treat $r\mathbb{Z}^{2n}$ as a subset of \mathbb{C}^n . A subset $G = \{p_0, \dots, p_k\}$ of $r\mathbb{Z}^{2n}$ with $k \geq 1$ is said to be a discrete segment in $r\mathbb{Z}^{2n}$ if there exists $j \in \{1, \dots, 2n\}$ and $z \in r\mathbb{Z}^{2n}$ such that

$$p_\ell = z + \ell(re_j), \quad 0 \leq \ell \leq k$$

where e_j is the standard j^{th} basis vector of \mathbb{R}^{2n} . In this setting, we say that p_0 and p_k are the endpoints of G . Also, we define the length

$|G|$ of G to be $|G| = k$. Let $\nu = (r\nu_1, \dots, r\nu_{2n})$ and $\nu' = (r\nu'_1, \dots, r\nu'_{2n})$ be elements of $r\mathbb{Z}^{2n}$ where $\nu \neq \nu'$. We can enumerate the integers $\{j : \nu_j \neq \nu'_j, 1 \leq j \leq 2n\}$ as j_1, \dots, j_m in ascending order, so that $j_1 < \dots < j_m$ when $m > 1$. Set $z_0(\nu, \nu') = \nu$, and inductively define $z_t(\nu, \nu') = z_{t-1}(\nu, \nu') + (\nu'_{j_t} - \nu_{j_t})(re_{j_t})$ for $t \in \{1, \dots, m\}$. Note that $z_m(\nu, \nu') = \nu'$. Let $G_t(\nu, \nu')$ be the discrete segment in $r\mathbb{Z}^{2n}$ which has $z_{t-1}(\nu, \nu')$ and $z_t(\nu, \nu')$ as its endpoints. The union of the discrete segments $G_1(\nu, \nu'), \dots, G_m(\nu, \nu')$ will be denoted by $\Gamma(\nu, \nu')$. We call $\Gamma(\nu, \nu')$ the discrete path in $r\mathbb{Z}^{2n}$ from ν to ν' . Furthermore, we define the length $|\Gamma(\nu, \nu')|$ of $\Gamma(\nu, \nu')$ to be $|G_1(\nu, \nu')| + \dots + |G_m(\nu, \nu')|$. That is, the length of $\Gamma(\nu, \nu')$ is just the sum of the lengths of the discrete segments which make up $\Gamma(\nu, \nu')$. In the case $\nu = \nu'$, we define the discrete path from ν to ν to be the singleton set $\Gamma(\nu, \nu) = \{\nu\}$.

Lemma 3.4. *If $w \in A_{p,3r}$ then there exists $C > 0$ independent of $\nu, \nu' \in r\mathbb{Z}^{2n}$ such that*

$$\frac{w(Q_r(\nu))}{w(Q_r(\nu'))} \leq C^{|\nu - \nu'|}$$

Proof. Enumerate the elements in $\Gamma(\nu, \nu')$ as a_0, a_1, \dots, a_k where $a_0 = \nu$, $a_k = \nu'$, $k = |\Gamma(\nu, \nu')|$, and

$$Q_r(a_{j-1}) \subseteq 3Q_r(a_j)$$

for each $j \in \{1, \dots, k\}$. Then by Lemma 3.2, there exists $C > 0$ where

$$\begin{aligned} \frac{w(Q_r(\nu))}{w(Q_r(\nu'))} &= \prod_{j=1}^k \frac{w(Q_r(a_{j-1}))}{w(Q_r(a_j))} \\ &\leq \prod_{j=1}^k \frac{w(3Q_r(a_j))}{w(Q_r(a_j))} \\ &\leq C^{|\Gamma(\nu, \nu')|}. \end{aligned}$$

However, an easy application of the Cauchy-Schwarz inequality tells us that $|\Gamma(\nu, \nu')| \leq \frac{(2n)^{\frac{1}{2}}|\nu - \nu'|}{r}$, which completes the proof. \square

We will now prove Theorem 3.1.

Proof of Theorem 3.1: We will first prove that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$. Then since trivially $(d) \Rightarrow (a)$, we will complete the proof by showing that $(b) \Rightarrow (d)$.

Let $r' = \frac{1}{3}r$. To show that (a) \Rightarrow (b), we have:

$$\begin{aligned}
& \|H_\alpha f\|_{L^p(\mathbb{C}^n, w dv)}^p \\
& \leq \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} e^{-\frac{\alpha}{2}|z-u|^2} |f(u)| dv(u) \right)^p w(z) dv(z) \\
& = \sum_{\nu \in r'\mathbb{Z}^{2n}} \int_{Q_{r'}(\nu)} \left(\sum_{\nu' \in r'\mathbb{Z}^{2n}} \int_{Q_{r'}(\nu')} e^{-\frac{\alpha}{2}|z-u|^2} |f(u)| dv(u) \right)^p w(z) dv(z) \\
& \leq C \sum_{\nu \in r'\mathbb{Z}^{2n}} \int_{Q_{r'}(\nu)} \left(\sum_{\nu' \in r'\mathbb{Z}^{2n}} e^{-\frac{\alpha}{4}|\nu-\nu'|^2} \int_{Q_{r'}(\nu')} |f(u)| dv(u) \right)^p w(z) dv(z) \\
& = \sum_{\nu \in r'\mathbb{Z}^{2n}} w(Q_{r'}(\nu)) \left(\sum_{\nu' \in r'\mathbb{Z}^{2n}} e^{-\frac{\alpha}{4}|\nu-\nu'|^2} \int_{Q_{r'}(\nu')} |f(u)| dv(u) \right)^p
\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}
& \sum_{\nu \in r'\mathbb{Z}^{2n}} w(Q_{r'}(\nu)) \left(\sum_{\nu' \in r'\mathbb{Z}^{2n}} e^{-\frac{\alpha}{4}|\nu-\nu'|^2} \int_{Q_{r'}(\nu')} |f(u)| dv(u) \right)^p \\
& \leq C \sum_{\nu \in r'\mathbb{Z}^{2n}} w(Q_{r'}(\nu)) \sum_{\nu' \in r'\mathbb{Z}^{2n}} e^{-\frac{p\alpha}{8}|\nu-\nu'|^2} \left(\int_{Q_{r'}(\nu')} |f(u)| dv(u) \right)^p
\end{aligned} \tag{3.2}$$

However, since $w \in A_{p,3r'}$, Lemmas 3.3 and 3.4 give us that

$$\begin{aligned}
w(Q_{r'}(\nu)) \left(\int_{Q_{r'}(\nu')} |f| dv \right)^p & \leq C \frac{w(Q_{r'}(\nu))}{w(Q_{r'}(\nu'))} \int_{Q_{r'}(\nu')} |f|^p w dv \\
& \leq C^{|\nu-\nu'|+1} \int_{Q_{r'}(\nu')} |f|^p w dv
\end{aligned} \tag{3.3}$$

Plugging (3.3) into (3.2) and switching the order of summation, we have that

$$\begin{aligned}
 & \sum_{\nu \in r' \mathbb{Z}^{2n}} w(Q_{r'}(\nu)) \sum_{\nu' \in r' \mathbb{Z}^{2n}} e^{-\frac{p\alpha}{8}|\nu-\nu'|^2} \left(\int_{Q_{r'}(\nu')} |f(u)| dv(u) \right)^p \\
 & \leq \sum_{\nu \in r' \mathbb{Z}^{2n}} \sum_{\nu' \in r' \mathbb{Z}^{2n}} C^{|\nu-\nu'|+1} e^{-\frac{p\alpha}{8}|\nu-\nu'|^2} \int_{Q_{r'}(\nu')} |f|^p w dv \\
 & = \sum_{\nu' \in r' \mathbb{Z}^{2n}} \int_{Q_{r'}(\nu')} |f|^p w dv \sum_{\nu \in r' \mathbb{Z}^{2n}} C^{|\nu-\nu'|+1} e^{-\frac{p\alpha}{8}|\nu-\nu'|^2} \\
 & \leq C \int_{\mathbb{C}^n} |f|^p w dv
 \end{aligned}$$

That (b) \Rightarrow (c) follows from a simple computation.

Let us now prove that (c) \Rightarrow (a). The proof will involve a modification of the proof of the corresponding result in [4] for the Hilbert transform on the weighted space $L^p(\mathbb{R}, w dx)$. Fix some cube Q with center z_0 and side length r_0 where $r_0 > 0$ is a small number to be determined. If

$$f(u) = w^{-\frac{1}{p-1}}(u) e^{\frac{\alpha}{2}|u|^2} e^{-i\alpha \text{Im}(z_0 \cdot u)} \chi_Q(u),$$

then

$$|P_\alpha f(z)| = \left(\frac{\alpha}{\pi} \right)^n \left| \int_Q e^{\alpha(z \cdot u)} e^{-\frac{\alpha}{2}|u|^2} e^{-i\alpha \text{Im}(z_0 \cdot u)} w^{-\frac{1}{p-1}}(u) dv(u) \right| \quad (3.4)$$

However,

$$\begin{aligned}
 e^{\alpha(z \cdot u)} &= |e^{\alpha(z \cdot u)}| e^{i\alpha \text{Im}(z \cdot u)} \\
 &= |e^{\alpha(z \cdot u)}| e^{i\alpha \text{Im}(z-z_0) \cdot (u-z_0)} e^{i\alpha \text{Im}(z_0 \cdot u)} e^{i\alpha \text{Im}(z-z_0) \cdot z_0}
 \end{aligned} \quad (3.5)$$

Plugging (3.5) into (3.4) gives

$$|P_\alpha f(z)| = \left(\frac{\alpha}{\pi} \right)^n e^{\frac{\alpha}{2}|z|^2} \left| \int_Q e^{-\frac{\alpha}{2}|z-u|^2} e^{i\alpha \text{Im}(z-z_0) \cdot (u-z_0)} w^{-\frac{1}{p-1}}(u) dv(u) \right|$$

Picking $r_0 > 0$ small enough, we get that $|1 - e^{i\alpha \text{Im}(z-z_0) \cdot (u-z_0)}| \leq \frac{1}{2}$ for all z and $u \in Q$, so writing $e^{i\alpha \text{Im}(z-z_0) \cdot (u-z_0)} = 1 - (1 - e^{i\alpha \text{Im}(z-z_0) \cdot (u-z_0)})$ and using the triangle inequality, we get that

$$\begin{aligned}
 |P_\alpha f(z)| &\geq \frac{1}{2} \left(\frac{\alpha}{\pi} \right)^n e^{\frac{\alpha}{2}|z|^2} \chi_Q(z) \int_Q e^{-\frac{\alpha}{2}|z-u|^2} w^{-\frac{1}{p-1}}(u) dv(u) \\
 &\geq C e^{\frac{\alpha}{2}|z|^2} \chi_Q(z) \int_Q w^{-\frac{1}{p-1}} dv.
 \end{aligned} \quad (3.6)$$

The boundedness of P_α on $\mathcal{L}_\alpha^p(w)$ applied to (3.6) now gives us that

$$w(Q) \left(\int_Q w^{-\frac{1}{p-1}} dv \right)^p \leq C \int_Q w^{-\frac{1}{p-1}} dv$$

which proves (a). Finally, the proof that (b) \Rightarrow (d) is similar to the proof that (c) \Rightarrow (a). \square

Remark 3.5. By Theorem 3.1 we have that the classes $A_{p,r}$ coincide for each $r > 0$. Thus, to emphasize this fact, we will denote the space $A_{p,r}$ by $A_p^{\text{restricted}}$.

Also, since $A_p^{\text{restricted}}$ is obviously independent of α , we have that P_{α_0} is bounded on $\mathcal{L}_{\alpha_0}^p(w)$ for some $\alpha_0 > 0$ if and only if P_α is bounded on $\mathcal{L}_\alpha^p(w)$ for all $\alpha > 0$.

Remark 3.6. The definition of $A_p^{\text{restricted}}$ can obviously be defined on \mathbb{R}^n for all $n \in \mathbb{N}$. Moreover, we also have that $A_p^{\text{restricted}}$ is the same as the class of weights w on \mathbb{R}^n where H_α is bounded on $L^p(\mathbb{R}^n, w dv)$ for any (or all) $\alpha > 0$.

We will now connect the class $A_p^{\text{restricted}}$ with an appropriate BMO type space. For $1 \leq p < \infty$, let BMO_r^p be the space of functions f on \mathbb{R}^n such that

$$\sup_{z \in \mathbb{R}^n} \frac{1}{v(B(z, r))} \int_{B(z, r)} |f - f_{B(z, r)}|^p dv < \infty$$

where $B(z, r)$ is a Euclidean ball of center $z \in \mathbb{R}^n$ and radius $r > 0$. It is easy to show that as a vector space, BMO_r^p is independent of $r > 0$, and so we will write BMO^p instead of BMO_r^p . It is also not hard to show that $\text{BMO}^p = \text{BA}^p + \text{BO}$ where $f \in \text{BO}$ if

$$\sup_{z \in \mathbb{R}^n} \omega_r(f)(z) < \infty$$

for some (or any) fixed $r > 0$ where $\omega_r(f)(z) = \sup_{w \in B(z, r)} |f(z) - f(w)|$

and $f \in \text{BA}^p$ if

$$\sup_{z \in \mathbb{R}^n} \frac{1}{v(B(z, r))} \int_{B(z, r)} |f|^p dv < \infty.$$

for some (or any) fixed $r > 0$. Note that both of these conditions are independent of $r > 0$. Also note that this decomposition is explicit.

In particular, if $f \in \text{BMO}^p$, then one can verify that $f_{B(\cdot, r)} \in \text{BO}$ and $f - f_{B(\cdot, r)} \in \text{BA}^p$ for any $r > 0$. Unlike in the classical BMO setting, note that the John-Nirenberg theorem is not true for the spaces BMO^p since the space BA^p depends on p . For more details about BMO^p (and for proofs of the above assertions) see [5], p. 3023.

However, similar to the classical BMO setting, one can show that $\log w \in \text{BMO}^1$ if $w \in A_p^{\text{restricted}}$, where the proof is identical to the proof in the classical A_p - BMO setting (see [9] p. 151.) It is also well known that in the classical setting, $e^{\delta f} \in A_p$ for $f \in \text{BMO}$ with $\delta > 0$ small enough (again see [9] p. 151). It would be interesting to know if any similar relationship between BMO^p and $A_p^{\text{restricted}}$ exists.

With Theorem 3.1 proved, we can now characterize invertible Toeplitz products on the Fock space. In fact, we will characterize bounded Toeplitz products $T_f T_{\bar{g}}$ when f, g are entire and as a consequence, as mentioned before, we will show that Sarason's conjecture is trivially true for the Fock space. First, for a function f on \mathbb{C}^n , let $\tilde{f}^{(\alpha)}$ be the Berezin transform of f given by

$$\tilde{f}^{(\alpha)}(z) = \left(\frac{\alpha}{\pi}\right)^n \int_{\mathbb{C}^n} e^{-\alpha|z-u|^2} f(u) dv(u).$$

Note that $\tilde{f}^{(\alpha)}$ can obviously be defined for a function f on \mathbb{R}^n . Moreover, for a function f on \mathbb{R}^n , note that $\tilde{f}^{(\alpha)}$ is just the convolution of f with the heat kernel $H(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\{-|x|^2/4t\}$ at time $t = \frac{1}{4\alpha}$.

Theorem 3.7. *Let $p > 1$ and let q be the conjugate exponent of p . If $f \in F_\alpha^p$ and $g \in F_\alpha^q$, then the following are equivalent:*

- (a) $T_f T_{\bar{g}}$ is bounded on F_α^p .
- (b) f and g satisfy

$$\sup_{z \in \mathbb{C}^n} \left(\widetilde{|f|^p}^{(\frac{\alpha p}{2})}(z) \right)^{\frac{1}{p}} \left(\widetilde{|g|^q}^{(\frac{\alpha q}{2})}(z) \right)^{\frac{1}{q}} < \infty.$$

Furthermore, if either of these are true then fg is identically constant, and if both f and g never vanish on \mathbb{C}^n , then $T_f T_{\bar{f}^{-1}}$ (and $T_{g^{-1}} T_{\bar{g}}$) is invertible on F_α^p .

Proof. We first prove that (a) \implies (b). Assume that $T_f T_{\bar{g}}$ is bounded on F_α^p . Since $\text{span}\{k_\eta : \eta \in \mathbb{C}^n\}$ is dense in F_α^p and F_α^q (see [13]), we have that

$$T_f T_{\bar{g}} k_z = \overline{g(z)} f k_z.$$

Moreover, it is easy to see that $|f(z)| \leq C \left(\widetilde{|f|^p}^{(\frac{\alpha p}{2})}(z) \right)^{\frac{1}{p}}$ for some $C > 0$ independent of f and z , so that

$$\begin{aligned} \sup_{z \in \mathbb{C}^n} |f(z)g(z)| &\leq \sup_{z \in \mathbb{C}^n} |g(z)| \left(\widetilde{|f|^p}^{(\frac{\alpha p}{2})}(z) \right)^{\frac{1}{p}} \\ &= \sup_{z \in \mathbb{C}^n} \|T_f T_{\bar{g}} k_z\|_{F_\alpha^p} \end{aligned} \quad (3.6)$$

that fg is identically a constant since $\|k_z\|_{F_\alpha^p} = 1$. Also, it is easy to see that either $f \equiv 0$ or $g \equiv 0$ if either f or g vanishes anywhere on \mathbb{C}^n . Thus, assume that both f and g never vanish on \mathbb{C}^n . Since $(T_f T_{\bar{g}})^* = T_g T_{\bar{f}}$ is bounded on $(F_\alpha^p)^* = F_\alpha^q$ (again see [13]), we also have that

$$\sup_{z \in \mathbb{C}^n} |f(z)| \left(\widetilde{|g|^q}^{(\frac{\alpha q}{2})}(z) \right)^{\frac{1}{q}} = \sup_{z \in \mathbb{C}^n} \|T_g T_{\bar{f}} k_z\|_{F_\alpha^q} \quad (3.7)$$

Combining (3.6) and (3.7) now gives us that

$$\sup_{z \in \mathbb{C}^n} \left(\widetilde{|f|^p}^{(\frac{\alpha p}{2})}(z) \right)^{\frac{1}{p}} \left(\widetilde{|g|^q}^{(\frac{\alpha q}{2})}(z) \right)^{\frac{1}{q}} < \infty.$$

Now we prove that $(b) \implies (a)$. If (b) is true, then again fg is identically constant, and if either f or g vanish anywhere on \mathbb{C}^n , then one of these functions is identically zero. Moreover, if (b) is true and both f and g never vanish on \mathbb{C}^n , then it is easy to see that $|f|^p \in A_p^{\text{restricted}}$, which means that $T_f T_{\bar{f}^{-1}}$ (and also $T_f T_{\bar{g}}$) is bounded on F_α^p .

Finally, assume that both f and g never vanish on \mathbb{C}^n . As in the proof of Theorem 2.1, it is easy to see that the inverse of $T_f T_{\bar{f}^{-1}}$ is T_ϕ where $\phi = \bar{f}/f$. \square

Remark 3.8. *Using ideas from the proof of Theorem 5.1 in Section 5, it is not difficult to see that the following are equivalent for any measurable f on \mathbb{C}^n :*

- (a) $\sup_{z \in \mathbb{C}^n} \left(\widetilde{|f|^p}^{(\alpha)}(z) \right)^{\frac{1}{p}} \left(\widetilde{|f|^{-q}}^{(\beta)}(z) \right)^{\frac{1}{q}} < C_{\alpha, \beta}$ for some $\alpha, \beta > 0$.
- (b) $\sup_{z \in \mathbb{C}^n} \left(\widetilde{|f|^p}^{(\alpha)}(z) \right)^{\frac{1}{p}} \left(\widetilde{|f|^{-q}}^{(\beta)}(z) \right)^{\frac{1}{q}} < C_{\alpha, \beta}$ for all $\alpha, \beta > 0$.
- (c) $w = |f|^p$ belongs to $A_p^{\text{restricted}}$.

This means that Theorem 3.5 can be strengthened considerably. In particular, even though the Berezin transforms in condition (b) of Theorem 3.5 might be difficult to compute or estimate directly, it is usually rather easy to check whether a weight w is in $A_p^{\text{restricted}}$ or not. Also note that if $w \in A_p^{\text{restricted}}$, then an easy application of Lemma 3.3 tells us that either $w \equiv 0$ a.e. or $w > 0$ a.e. For precisely these reasons, we will state the following corollary to Theorem 3.5:

Corollary 3.9. *Let f be any measurable function on \mathbb{C}^n where $f \not\equiv 0$ a.e. and where $w = |f|^p$ is in $A_p^{\text{restricted}}$. Then $T_f T_{f^{-1}}$ is bounded on \mathcal{L}_α^p (and in particular, bounded on F_α^p) for any $\alpha > 0$. Also, the same statement holds for $T_f T_{f^{-1}}$.*

Remark 3.10. *A typical example of an entire function f where $|f|^p \in A_p^{\text{restricted}}$ and f^{-1} is entire is $f(z) = e^{P(z)}$ where $P(z)$ is a linear polynomial. It would be interesting to know precisely which functions are of this type.*

Note however that if f is an entire function with $|f|^p \in A_p^{\text{restricted}}$ and f^{-1} is entire, then there exists constants C_1, C_2, C_3, C_4 where

$$C_1 e^{C_2|z|} \leq |f(z)| \leq C_3 e^{C_4|z|}$$

for any $z \in \mathbb{C}^n$. To see this, first note that by essentially the definition of BO , we have

$$|g(z)| \leq A + B|z|$$

for some constants $A, B \geq 0$ if $g \in BO$. Now since $\log |f| \in BMO^1$ and is subharmonic, we have that

$$\log |f|(z) \leq (\log |f|)_{B(z,1)} \leq C_1 + C_2|z|$$

since $(\log |f|)_{B(z,1)} \in BO$. Applying the same reasoning to $|f|^{-\frac{p}{p-1}}$ completes the proof.

4. CLASSES OF WEIGHTS

In this section, we will analyze the classes of weights relevant to the results of the previous sections.

First, for $p > 1$, define the invariant A_p class (which will be denoted by $A_p^{\text{inv.}}$) to be the class of all weights w on $\partial\mathbb{B}_n$ such that

$$\sup_{z \in \mathbb{B}_n} \{\widehat{w}(z)\} \widehat{\{w^{-\frac{1}{p-1}}(z)\}}^{p-1} < \infty.$$

Note that by definition, $A_p^{\text{inv.}}$ is Möbius invariant. For a definition and discussion of $A_\infty^{\text{inv.}}$ weights on $\partial\mathbb{D}$, see [22] and [23].

For $p = 2$, it is not difficult to show that A_2 and $A_2^{\text{inv.}}$ are the same classes. However, for general $p > 1$, $A_p^{\text{inv.}}$ is strictly larger than A_p (see [22, 23] for examples.) Also, for a discussion of $A_p^{\text{inv.}}$ weights on \mathbb{R} for $1 < p < \infty$, see [10].

With this in mind, one can similarly define $B_{p,\gamma}^{\text{inv.}}$ to be the class of all weights w on \mathbb{B}_n where

$$\sup_{z \in \mathbb{B}_n} \{B_\gamma(w)(z)\} \{B_\gamma(w^{-\frac{1}{p-1}})(z)\}^{p-1} < \infty.$$

Note that $B_{p,\gamma}^{\text{inv.}}$ is also Möbius invariant

We can also describe $B_{p,\gamma}$ in terms of the Berezin transform. In particular, we have:

Proposition 4.1. *A weight w on \mathbb{B}_n is in $B_{p,\gamma}$ if and only if*

$$\|w\|_{B_{p,\gamma}^{\text{Ber.}}} = \sup_{z \in \mathbb{B}_n} \{B_\gamma(w|k_z^{p-2}|)(z)\} \{B_\gamma(w^{-\frac{1}{p-1}}|k_z^{q-2}|)(z)\}^{p-1} < \infty.$$

In particular, there exists a constant C independent of w where

$$\frac{1}{C} \|w\|_{B_{p,\gamma}} \leq \|w\|_{B_{p,\gamma}^{\text{Ber.}}} \leq C \|w\|_{B_{p,\gamma}}^{\max\{2, \frac{q}{p}+1\}}.$$

Note that this proposition tells us that $B_{p,\gamma}^{\text{inv.}} = B_{p,\gamma}$ when $p = 2$.

If we define $w_\zeta(z) = (1 - |z|^2)^\zeta$ for $\zeta \in \mathbb{R}$, then a messy but elementary application of the Rudin-Forelli estimates (see [26]) gives us the following two propositions:

Proposition 4.2. *$w_\zeta \in B_{p,\gamma}$ if and only if $-1 - \gamma < \zeta < (1 + \gamma)(p - 1)$.*

Proposition 4.3. *$w_\zeta \in B_{p,\gamma}^{\text{inv.}}$ if and only if*

- (1) $-1 - \gamma < \zeta < (1 + \gamma)(p - 1)$, and
- (2) $-(p - 1)(n + 1 + \gamma) < \zeta < n + 1 + \gamma$.

These two propositions tell us that the classes $B_{p,\gamma}^{\text{inv.}}$ and $B_{p,\gamma}$ do not coincide when either $p > 2 + \frac{n}{1+\gamma}$ or $p < 1 + \frac{1+\gamma}{n+1+\gamma}$. However, it is unlikely that $B_{p,\gamma}^{\text{inv.}}$ and $B_{p,\gamma}$ coincide for any $p > 1$, $n \geq 1$, and $\gamma > -1$.

Also, we have the following analog of Proposition 4.1 for $\partial\mathbb{B}_n$:

Proposition 4.4. *A weight w on $\partial\mathbb{B}_n$ is in A_p if and only if w satisfies*

$$\|w\|_{A_p^{\text{Poi.}}} = \sup_{z \in \mathbb{B}_n} \{\widehat{w|k_z|^{p-2}}(z)\} \{\widehat{w^{-\frac{1}{p-1}}|k_z|^{q-2}}(z)\}^{p-1} < \infty.$$

In fact, there exists a constant C independent of w where

$$\frac{1}{C} \|w\|_{A_p} \leq \|w\|_{A_p^{\text{Poi.}}} \leq C \|w\|_{A_p}^{\max\{2, \frac{q}{p}+1\}}.$$

Here, $k_z(w) = \frac{(1-|z|^2)^{n/2}}{(1-w\bar{z})^n}$ is the normalized reproducing kernel of $H^2(\partial\mathbb{B}_n)$.

We will defer the proof of Propositions 4.1 and 4.4 until the last section since the proof uses ideas found there. It should be noted that Propositions 4.1 and 4.4 are surprisingly *not* true in the \mathbb{R}^n setting when $n \geq 2$. In particular, if $w(x) = |x|^\alpha$, then $w \in A_2$ if and only if $|\alpha| < n$, whereas $w \notin A_2^{\text{poi}}$ if $\alpha \geq 1$ (see [17]).

When $p = 2$, Proposition 4.4 was proven to be sharp in [10] for \mathbb{R} . In particular, if $-1 < \alpha < 1/2$ and if

$$w = \begin{cases} 1 & \text{for } x \in [0, 1)^c \\ (1 - \alpha)^n & \text{for } x \in (1/2^{n+1}, 1/2^n] \end{cases} \quad (4.1)$$

then $\|w\|_{A_2} \approx (1 - 2\alpha)^{-1}$, while $\|w\|_{A_2^{\text{inv}}} \approx (1 - 2\alpha)^{-2}$. Since this example can easily be extended to $\partial\mathbb{D}$, it would be interesting to know if some example similar to (4.1) can be cooked up for the unit disk or the unit ball.

It would also be interesting to know if one can use Bergman balls of a fixed radius when defining the class $B_{p,\gamma}$ in equation (2.1). In other words, given any $r > 0$, is a weight $w \in B_{p,\gamma}$ if and only if

$$\sup_{z \in \mathbb{B}_n} \left(\frac{1}{v_\gamma(D(z, r))} \int_{D(z, r)} w \, dv_\gamma \right) \left(\frac{1}{v_\gamma(D(z, r))} \int_{D(z, r)} w^{-\frac{1}{p-1}} \, dv_\gamma \right)^{p-1} < C_r$$

for some $C_r > 0$, where here $D(z, r) \subseteq \mathbb{B}_n$ is a ball with respect to the Bergman metric with center z and radius r ? Because of the lack of decay provided by the normalized reproducing kernel, it is easy to see that an argument like the one in the proof of Theorem 3.1 combined with Proposition 4.1 can not be used to prove this. However, it is not clear if this is simply due to the inefficiency of this specific approach.

It should be remarked that the Muckenhoupt A_p class on \mathbb{R}^n coincides with the class of all weights w on \mathbb{R}^n such that

$$\|w\|_{A_p^{\text{heat}}} = \sup_{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}_+} (\widetilde{w^{(\alpha)}}(x)) \left(\widetilde{w^{-\frac{1}{p-1}}}(x) \right)^{p-1} < \infty$$

(this was proven in [18] for $n = 1$, but the proof can easily be extended to the $n > 1$ case.) Moreover, the natural “norms” defined by the corresponding supremums are equivalent.

On the other hand, an argument that is similar to (but easier than) the proof of Theorem 3.1 tells us that $A_p^{\text{restricted}}$ coincides with the class of

all weights w on \mathbb{R}^n where for each $\alpha, \beta > 0$, there is some $C_{\alpha, \beta} < \infty$ such that

$$\sup_{x \in \mathbb{R}^n} (\tilde{w}^{(\alpha)}(x)) \left(\widetilde{w^{-\frac{1}{p-1}}}^{(\beta)}(x) \right)^{p-1} < C_{\alpha, \beta}. \quad (4.2)$$

Unfortunately, the argument gives no relationship between (4.2) for fixed α, β and the $A_{p,r}$ “norm” of a weight for a fixed r , though trivially there exists $C_{r, \alpha, \beta}$ where

$$\|w\|_{A_{p,r}} \leq C_{r, \alpha, \beta} \sup_{x \in \mathbb{R}^n} (\tilde{w}^{(\alpha)}(x)) \left(\widetilde{w^{-\frac{1}{p-1}}}^{(\beta)}(x) \right)^{p-1}.$$

We will end our discussion on A_p and $B_{p,\gamma}$ weights by comparing one last property of A_p and $B_{p,\gamma}$ weights. Recall that Coifmann and Fefferman proved in [4] that

$$A_p = \bigcup_{1 < q < p} A_q$$

if $p > 1$. Note that one side of this equality holds trivially by Hölder’s inequality. Using the proof Lemma 5.9 in Section 5, it is not difficult to see that

$$L_a^p(\mathbb{D}, dA_\gamma) \cap B_{p,\gamma} \subseteq \bigcup_{1 < q < p} B_{q,\gamma} \quad (4.3)$$

When $\gamma = 0$, A. Borichev generalized (4.3) and proved (among other things) that

$$\mathcal{ES} \cap B_{p,\gamma} \subseteq \bigcup_{1 < q < p} B_{q,\gamma}$$

where \mathcal{ES} is the class of all functions e^u for u subharmonic on \mathbb{D} (see [2]). Furthermore, it was shown in [2] that if \mathcal{S} is the class of non-negative subharmonic functions on \mathbb{D} , then

$$\mathcal{S} \cap B_{p,\gamma} \not\subseteq \bigcup_{1 < q < p} B_{q,\gamma}.$$

Given these results, it would be interesting to know if the results in [2], can be extended to general $\gamma > -1$ and $n > 1$, or if (4.3) is true for $n > 1$.

Finally, note that we have

$$A_p^{\text{restricted}} = \bigcup_{1 < q < p} A_q^{\text{restricted}}. \quad (4.4)$$

The proof is almost the same as the proof of Corollary 7.6 in [9]. In particular, an easy application of Lemma 3.3 gives us the following:

Proposition 4.5. *Let Q_r be a cube of side length $r > 0$ and let $w \in A_p^{\text{restricted}}$. If $0 < \delta < 1$, then there exists $\epsilon = \epsilon_{r,\delta}$ with $0 < \epsilon < 1$ such that $w(S) \leq \epsilon w(Q_r)$ whenever $S \subseteq Q_r$ and $v(S) \leq \delta v(Q_r)$.*

Using this proposition, one can use the proof of Theorem 7.4 in [9] to show that w satisfies the following “reverse Hölder inequality:” if $w \in A_p^{\text{restricted}}$, then there exists $C > 0$ and $\epsilon > 0$ (both depending on w and r , but not on the specific cube Q_r used) such that

$$\left(\frac{1}{v(Q_r)} \int_{Q_r} w^{1+\epsilon} dv \right)^{\frac{1}{1+\epsilon}} \leq \frac{C}{v(Q_r)} \int_{Q_r} w dv. \quad (4.5)$$

It is then easy to prove (4.4) using (4.5).

5. A “REVERSE HÖLDER INEQUALITY” ON \mathbb{D} .

In this last section, we will provide a proof of Theorem 2.1 for the disk \mathbb{D} by extending the ideas of [20, 21] from the $p = 2$ case to the general $p > 1$ case. In particular, we will prove the following “reverse Hölder inequality:”

Theorem 5.1. *Let $f \in L_a^p(\mathbb{D}, dA_\gamma)$ and $f^{-1} \in L_a^q(\mathbb{D}, dA_\gamma)$ satisfy*

$$\sup_{z \in \mathbb{D}} \left\{ B_\gamma \left(|f k_z^{1-2/p}|^p \right) (z) \right\}^{\frac{1}{p}} \left\{ B_\gamma \left(|f^{-1} k_z^{1-2/q}|^q \right) (z) \right\}^{\frac{1}{q}} < \infty. \quad (5.1)$$

Then there exists $\epsilon > 0$ such that

$$\sup_{z \in \mathbb{D}} \left\{ B_\gamma \left(|f k_z^{1-2/p}|^{p+\epsilon} \right) (z) \right\}^{\frac{1}{p+\epsilon}} \left\{ B_\gamma \left(|f^{-1} k_z^{1-2/q}|^{q+\epsilon} \right) (z) \right\}^{\frac{1}{q+\epsilon}} < \infty. \quad (5.2)$$

Once this is proved, Theorem 1.2 of [15] will give us that $T_f T_{\overline{f^{-1}}}$ is bounded on $L_a^p(\mathbb{D}, dA_\gamma)$. Easy arguments from Section 2 will then complete the proof of Theorem 2.1 for $n = 1$.

When $p = 2$, condition (5.2) is Möbius invariant, so that it is only necessary to prove that (5.1) implies (5.2) when $z = 0$ in (5.2) (which was done in [20, 21].) In other words, it is proven in [20, 21] that if both $f, f^{-1} \in L_a^2(\mathbb{D}, dA_\gamma)$ satisfy

$$\sup_{z \in \mathbb{D}} \left\{ B_\gamma \left(|f|^2 \right) (z) \right\}^{\frac{1}{2}} \left\{ B_\gamma \left(|f|^{-2} \right) (z) \right\}^{\frac{1}{2}} < \infty,$$

then there exists $\epsilon > 0$ and $C > 0$ such that

$$\left(\int_{\mathbb{D}} |f|^{2+\epsilon} dA_\gamma \right)^{\frac{1}{2+\epsilon}} \leq C \left(\int_{\mathbb{D}} |f|^2 dA_\gamma \right)^{\frac{1}{2}}. \quad (5.3)$$

When $p \neq 2$, condition (5.2) is not necessarily Möbius invariant, which means that it is not enough to just verify (5.3) (where p replaces 2.)

To prove Theorem 5.1, we will decompose \mathbb{D} into convenient Carleson squares using the “Bergman tree” of [1]. We will then run a Calderon-Zygmund decomposition on each of these Carleson squares to prove a reverse Hölder type inequality on each of these Carleson squares that is similar to (5.3). This will allow us to prove that f satisfies an “ A_∞ type” condition with respect these Carleson squares if f satisfies (5.1). The decay provided by the normalized Bergman kernel, combined with this “ A_∞ type” condition, will then allow us to prove Theorem 5.1

We will now go through the details of the proof of Theorem 5.1. In what follows, we will use the notation $A \approx B$ for two quantities A and B if there exists $C > 0$ depending only on γ, n , and p where

$$\frac{1}{C}A \leq B \leq CA.$$

The notation $A \lesssim B$ and $A \gtrsim B$ will have similar meanings. For any $0 < h \leq 1$ and $0 \leq \theta < 2\pi$, let $S_{h,\theta} \subseteq \mathbb{D}$ denote the Carleson square defined by

$$S_{h,\theta} = \{re^{it} : 1-h \leq r < 1, \theta \leq t < \theta+h\}$$

and let

$$T_{h,\theta} = \{re^{it} : 1-h \leq r < 1 - \frac{h}{2}, \theta \leq t < \theta+h\}$$

denote the “bottom half” of the Carleson square $S_{h,\theta}$. Here we will only be interested in Carleson and bottom half Carleson squares of the form $S_{h,\theta}$ where $h = 2^{-n}$ and $\theta = 2\pi(k2^{-n})$ for $n = 0, 1, 2, \dots$ and $k = 0, 1, \dots, 2^n - 1$.

Let us now introduce the “Bergmann tree” of Arcozzi, Rochberg, and Sawyer for \mathbb{D} from [1]. Let \mathcal{D} be the index set defined by

$$\mathcal{D} = \{(n, k) : n = 0, 1, 2, \dots \text{ and } k = 0, 1, \dots, 2^n - 1\}.$$

We call $o = (0, 1)$ the root of \mathcal{D} . We give \mathcal{D} a partial ordering by declaring $\eta \leq \beta$ if $S_\beta \subseteq S_\eta$, and call \mathcal{D} with this partial ordering the Bergman tree. Note that this partial ordering means that $S_o \leq S_\beta$ for every $\beta \in \mathcal{D}$. Also, we will let c_β denote the center (radially and angularly) of T_β and let $d(\beta) = n$ if $\beta = (n, k)$. Moreover, if $\beta \leq \beta'$

with $d(\beta) = d(\beta') - 1$ then we say β' is a child of β . Clearly each $\beta \in \mathcal{D}$ has only two children. Note that by definition we have that

$$S_\eta = \bigcup_{\beta \geq \eta} T_\beta.$$

If $z, w \in \mathbb{D}$ where $z = re^{i\theta}$, $w = se^{i\vartheta}$, and $0 \leq \theta, \vartheta < 2\pi$, then it is easy to see that

$$|1 - z\bar{w}|^2 = (1 - rs)^2 + 4rs \sin^2 \left(\frac{\theta - \vartheta}{2} \right). \quad (5.4)$$

Thus, there exists $R > 0$ independent of $\beta \in \mathcal{D}$ such that

$$D(c_\beta, 1/R) \subseteq T_\beta \subseteq D(c_\beta, R) \quad (5.5)$$

where $D(z, r)$ is a Bergman disk of radius r and center z . Also, it is not difficult to see that

$$A_\gamma(T_\beta) \approx A_\gamma(S_\beta) \approx 2^{-d(\beta)(2+\gamma)}$$

for each $\beta \in \mathcal{D}$.

Given any S_β with $\beta \in \mathcal{D}$, we can form dyadic partitions of S_β by dyadically bisecting S_β in the angular and radial direction. Any subset $Q \subset S_\beta$ formed in this way will be called a dyadic subrectangle of S_β . Note that since $\mathbb{D} = S_o$, the “dyadic rectangles” of [20, 21] are dyadic subrectangles of \mathbb{D} according to our definition. In particular, any dyadic subrectangle of \mathbb{D} can be written in the form

$$Q_{n,m,k} = \{re^{i\theta} : (m-1)2^{-n} \leq r < m2^{-n} \text{ and } (k-1)2^{-n+1}\pi \leq \theta < k2^{-n+1}\pi\}$$

where k, m , and n are positive integers such that $m, k \leq 2^n$. Also, the center of $Q = Q_{n,m,k}$ is the point $z_Q = (m - \frac{1}{2})2^{-n}e^{i\vartheta}$ with $\vartheta = (k - \frac{1}{2})2^{1-n}\pi$. Throughout this section we will use z_Q to denote the center (angularly and radially) of a dyadic subrectangle of \mathbb{D} , whereas c_β will denote the center of T_β for $\beta \in \mathcal{D}$.

Lemma 5.2. *Let $f \in L_a^p(\mathbb{D}, dA_\gamma)$ satisfy (5.1) and let $R > 0$. Then there exists $C_R > 0$ such that*

$$\frac{1}{C_R} \leq \frac{|f(z)|}{|f(w)|} \leq C_R$$

whenever $z \in D(w, R)$.

Proof. The proof is very similar to the proof of Lemma 4.3 in [21], though we include it for the sake of completeness. According to lemma 4.30 in [25], there exists $C > 0$ depending on n, p, R , and γ such that

$$\frac{1}{C}(1 - |w|^2)^{\left(\frac{2}{p}-1\right)\left(\frac{2+\gamma}{2}\right)} \leq |k_w^{1-\frac{2}{p}}(z)| \leq C(1 - |w|^2)^{\left(\frac{2}{p}-1\right)\left(\frac{2+\gamma}{2}\right)}$$

whenever $z \in D(w, R)$,

For $z \in D(w, R)$, let $z = \varphi_w(u)$ with $u \in D(0, R)$. Then we have that

$$\begin{aligned} |f(z)| &\leq C(1 - |w|^2)^{\left(1-\frac{2}{p}\right)\left(\frac{2+\gamma}{2}\right)} |f(\varphi_w(u))| |k_w^{1-\frac{2}{p}}(\varphi_w(u))| \\ &\leq C(1 - |w|^2)^{\left(1-\frac{2}{p}\right)\left(\frac{2+\gamma}{2}\right)} \{B_\gamma(|f k_w^{1-\frac{2}{p}}|^p)(w)\}^{\frac{1}{p}}. \end{aligned}$$

Similarly, for f^{-1} we have that

$$\frac{1}{|f(w)|} \leq C(1 - |w|^2)^{\left(1-\frac{2}{q}\right)\left(\frac{2+\gamma}{2}\right)} \{B_\gamma(|f^{-1} k_w^{1-\frac{2}{q}}|^q)(w)\}^{\frac{1}{q}}$$

which means that

$$\frac{|f(z)|}{|f(w)|} \leq C \{B_\gamma(|f k_w^{1-\frac{2}{p}}|^p)(w)\}^{\frac{1}{p}} \{B_\gamma(|f^{-1} k_w^{1-\frac{2}{q}}|^q)(w)\}^{\frac{1}{q}} \leq C$$

where here C depends on R and the the supremum in (5.1). Replacing f by f^{-1} and p with q in the above argument now completes the proof. \square

The following two results were proven in [21].

Proposition 5.3. *For every dyadic subrectangle Q of \mathbb{D} and every $z \in Q$, we have that*

$$|k_{z_Q}(z)|^2 \gtrsim \frac{1}{(1 - |z_Q|^2)^{2+\gamma}}$$

Proposition 5.4. *There exists $R > 0$ such that $Q \subseteq D(z_Q, R)$ for every dyadic subrectangle Q of \mathbb{D} that has positive distance to $\partial\mathbb{D}$.*

Lemma 5.5. *Let $f \in L_a^p(\mathbb{D}, dA_\gamma)$ satisfy (5.1) and let $w = |f|^p$. Then for each $\beta \in \mathcal{D}$ and each dyadic subrectangle Q of S_β , we have that*

$$\left(\frac{1}{A_\gamma(Q)} \int_Q w dA_\gamma \right) \left(\frac{1}{A_\gamma(Q)} \int_Q w^{-\frac{1}{p-1}} dA_\gamma \right)^{p-1} \leq C \quad (5.6)$$

where C is independent of β and Q .

Proof. Clearly it is enough to show that there exists $C > 0$ independent of β and Q where

$$\left(\frac{1}{A_\gamma(Q)} \int_Q |f|^p dA_\gamma \right)^{\frac{1}{p}} \left(\frac{1}{A_\gamma(Q)} \int_Q |f|^{-q} dA_\gamma \right)^{\frac{1}{q}} \leq C.$$

First assume that $\beta = o$, so that $S_\beta = \mathbb{D}$. If $Q = \mathbb{D}$, then this follows immediately from (5.1). If $d(Q, \partial\mathbb{D}) > 0$ then the result immediately follows from Proposition 5.4 and Lemma 5.2. If $d(Q, \partial\mathbb{D}) = 0$ then the Lemma follows from Proposition 5.3 and the fact that $A_\gamma(Q) = 2^{3+2\gamma}|z_Q|^{1+\gamma}(1-|z_Q|)^{2+\gamma}$ (see [21].)

Now assume that $\beta \neq o$. Note that if we dyadically quadrisect S_β any number of times, then an easy induction shows that we either obtain one of three types of sets: $S_{\beta'}$ where $\beta' \geq \beta$, the left (or right) angular half of $T_{\beta'}$, or repeated quadrissection of the left (or right) angular half of $T_{\beta'}$. In particular, this tells us that any dyadic subrectangle Q of S_β is either $S_{\beta'}$ for some $\beta' \geq \beta$ or is contained in the hyperbolic disk $D(c_{\beta'}, R)$ where $\beta' \geq \beta$ and R is the constant from (5.5).

In the latter case, the Lemma follows immediately from Lemma 5.2. To finish the proof, we will show that Lemma 5.5 is true for each S_β . If $z \in S_\beta$ with $z = re^{i\theta}$ where $0 \leq \theta < 2\pi$, then by the definition of S_β we have that $|\theta - \vartheta| \leq 2^{-d(\beta)}$ where $c_\beta = se^{i\vartheta}$ with $0 \leq \vartheta < 2\pi$. Thus, since $(1 - |c_\beta|^2) \approx 2^{-d(\beta)}$, we have from (5.4) that

$$|k_{c_\beta}(z)| = \frac{(1 - |c_\beta|^2)^{\frac{2+\gamma}{2}}}{|1 - z\bar{c}_\beta|^{2+\gamma}} \gtrsim \frac{1}{A_\gamma(S_\beta)^{1/2}}$$

which tells us that

$$\begin{aligned} \{B_\gamma(|fk_{c_\beta}^{1-\frac{2}{p}}|^p)(c_\beta)\}^{\frac{1}{p}} &= \left(\int_{\mathbb{D}} |fk_{c_\beta}|^p dA_\gamma \right)^{\frac{1}{p}} \\ &\geq \left(\int_{S_\beta} |fk_{c_\beta}|^p dA_\gamma \right)^{\frac{1}{p}} \\ &\gtrsim \frac{1}{A_\gamma(S_\beta)^{1/2}} \left(\int_{S_\beta} |f|^p dA_\gamma \right)^{\frac{1}{p}}. \end{aligned}$$

Switching f with $\frac{1}{f}$, and switching p with q , now completes the proof. \square

The proof of the following is a standard application of Lemma 5.5 (and is very similar to the proof of Lemma 4.6 of [21]). The proof will therefore be omitted.

Lemma 5.6. *Let $f \in L_a^p(\mathbb{D}, dA_\gamma)$ satisfy (5.1) and let C_1 be the constant in Lemma 5.5. If $w = |f|^p$ and if $\delta = 1 - \frac{1}{2^p C_1}$, then we have*

$$w(E) \leq \delta w(Q)$$

whenever E is a subset of a dyadic subrectangle Q of any S_β where $A_\gamma(E) \leq \frac{1}{2}A_\gamma(Q)$.

Now, suppose that we have a dyadic subrectangle Q of S_β for some $\beta \in \mathcal{D}$. If Q is formed from $k \geq 1$ repeated dyadic quadrisections of S_β , then we define the double $2Q$ of Q to be the unique dyadic subrectangle of S_β formed by $k - 1$ repeated dyadic quadrisections of S_β that also contains Q . We will now establish a doubling property that extends Proposition 4.9 of [21].

Lemma 5.7. *For any $\beta \in \mathcal{D}$ and any dyadic subrectangle $Q \subsetneq S_\beta$, we have that $A_\gamma(2Q) \lesssim A_\gamma(Q)$.*

Proof. If Q is a dyadic subrectangle of \mathbb{D} , then this was proven in Proposition 4.9 of [21], so assume that Q is a dyadic subrectangle of S_β with $d(\beta) \geq 1$.

As stated in the proof of Lemma 5.5, repeated quadrisection of S_β gives us one of the following three sets: $S_{\beta'}$ where $\beta' \geq \beta$, the left (or right) angular half of $T_{\beta'}$, or the repeated quadrisection of the left (or right) angular half of $T_{\beta'}$. However, since $A_\gamma(S_\beta) \approx A_\gamma(T_\beta) \approx 2^{-d(\beta)(2+\gamma)}$, it is easy to see that $A_\gamma(2Q) \leq CA_\gamma(Q)$ for either of these cases, where $C > 0$ is independent of Q . \square

Lemma 5.8. *Let $f \in L_a^p(\mathbb{D}, dA_\gamma)$ satisfy (5.1). Also, let $\tilde{C} > 0$ be the constant in Lemma 5.7 and let δ be the constant from Lemma 5.6. If $\beta \in \mathcal{D}$, then for any dyadic subrectangle Q of S_β (including S_β itself), we have that*

$$\left(\frac{1}{A_\gamma(Q)} \int_Q w^{1+\epsilon} dA_\gamma \right)^{\frac{1}{1+\epsilon}} \leq \left(1 + \frac{(2\tilde{C})^\epsilon}{1 - (2\tilde{C})^\epsilon \delta} \right)^{\frac{1}{1+\epsilon}} \frac{1}{A_\gamma(Q)} \int_Q w dA_\gamma$$

whenever $(2\tilde{C})^\epsilon \delta < 1$.

Proof. Using Lemmas 5.6 and 5.7, the proof is identical to the proof of Theorem 7.4 in [9]. \square

Lemma 5.9. *Let $f \in L_a^p(\mathbb{D}, dA_\gamma)$ satisfy (5.1) and let $w = |f|^p$. Then for any $\beta \in \mathcal{D}$, any $E \subset S_\beta$, and small enough ϵ , we have*

$$\int_E w^{1+\epsilon} dA_\gamma \leq C \left(\int_{S_\beta} w^{1+\epsilon} dA_\gamma \right) \left(\frac{A_\gamma(E)}{A_\gamma(S_\beta)} \right)^{\frac{\epsilon}{1+\epsilon}}$$

where C is independent of E and β .

Proof. The proof is similar to the proof of Corollary 7.6 of [9], but requires a somewhat careful tracking of the constants involved. Let $\beta \in \mathcal{D}$ and let Q be any dyadic subrectangle of S_β . By Lemma 5.8,

$$\left(\frac{1}{A_\gamma(Q)} \int_Q w^{1+\epsilon_1} dA_\gamma \right)^{\frac{1}{1+\epsilon_1}} \leq \left(1 + \frac{(2\tilde{C})^{\epsilon_1}}{1 - (2\tilde{C})^{\epsilon_1} \delta} \right)^{\frac{1}{1+\epsilon_1}} \frac{1}{A_\gamma(Q)} \int_Q w dA_\gamma \quad (5.7)$$

whenever $(2\tilde{C})^{\epsilon_1} \delta < 1$ where $\delta = 1 - \frac{1}{2^p C_1}$ and C_1 is the constant in Lemma 5.5.

Similarly, since $w^{-\frac{1}{p-1}}$ satisfies the conclusion of Lemma 5.5 with “ A_q norm” C_1^{q-1} , we have that

$$\begin{aligned} \left(\frac{1}{A_\gamma(Q)} \int_Q w^{-(1+\epsilon_1)(\frac{1}{p-1})} dA_\gamma \right)^{\frac{1}{1+\epsilon_1}} \\ \leq \left(1 + \frac{(2\tilde{C})^{\epsilon_1}}{1 - (2\tilde{C})^{\epsilon_1} \delta'} \right)^{\frac{1}{1+\epsilon_1}} \frac{1}{A_\gamma(Q)} \int_Q w^{-\frac{1}{p-1}} dA_\gamma \end{aligned} \quad (5.8)$$

whenever $(2\tilde{C})^{\epsilon_1} \delta' < 1$ where $\delta' = 1 - \frac{1}{2^q C_1^{q-1}}$.

Combining (5.6), (5.7), and (5.8), we have that

$$\begin{aligned} \left(\frac{1}{A_\gamma(Q)} \int_Q w^{1+\epsilon_1} dA_\gamma \right) \left(\frac{1}{A_\gamma(Q)} \int_Q w^{-(1+\epsilon_1)(\frac{1}{p-1})} dA_\gamma \right)^{p-1} \\ \leq C_1^{1+\epsilon_1} \left(1 + \frac{(2\tilde{C})^{\epsilon_1}}{1 - (2\tilde{C})^{\epsilon_1} \delta} \right) \left(1 + \frac{(2\tilde{C})^{\epsilon_1}}{1 - (2\tilde{C})^{\epsilon_1} \delta'} \right)^{p-1} \end{aligned} \quad (5.9)$$

which means that $w^{1+\epsilon_1}$ satisfies the conclusion of Lemma 5.5 (for small enough ϵ_1) with “ A_p norm”

$$C_{1,\epsilon_1} = C_1^{1+\epsilon_1} \left(1 + \frac{(2\tilde{C})^{\epsilon_1}}{1 - (2\tilde{C})^{\epsilon_1} \delta} \right) \left(1 + \frac{(2\tilde{C})^{\epsilon_1}}{1 - (2\tilde{C})^{\epsilon_1} \delta'} \right)^{p-1}.$$

Moreover, (5.9) implies that Lemma 5.6 holds for $w^{1+\epsilon_1}$ with constant $\delta_{\epsilon_1} = 1 - \frac{1}{2^p C_{1,\epsilon_1}}$, and so another application of Lemma 5.8 with $Q = S_\beta$

gives us that

$$\begin{aligned} \left(\frac{1}{A_\gamma(S_\beta)} \int_{S_\beta} w^{(1+\epsilon_1)(1+\epsilon_2)} dA_\gamma \right)^{\frac{1}{1+\epsilon_2}} \\ \leq \left(1 + \frac{(2\tilde{C})^{\epsilon_2}}{1 - (2\tilde{C})^{\epsilon_2}\delta_{\epsilon_1}} \right)^{\frac{1}{1+\epsilon_2}} \frac{1}{A_\gamma(S_\beta)} \int_{S_\beta} w^{1+\epsilon_1} dA_\gamma \end{aligned} \quad (5.10)$$

so long as $\epsilon_2 > 0$ is chosen small enough to make $(2\tilde{C})^{\epsilon_2}\delta_{\epsilon_1} < 1$.

Finally, setting $\epsilon = \epsilon_2 = \epsilon_1$ where ϵ is chosen small enough and using (5.10) and Hölder's inequality, we have

$$\begin{aligned} w^{1+\epsilon}(E) &= \int_{S_\beta} \chi_E w^{1+\epsilon} dA_\gamma \\ &\leq \left(w^{(1+\epsilon)(1+\epsilon)}(S_\beta) \right)^{\frac{1}{1+\epsilon}} A_\gamma(E)^{\frac{\epsilon}{1+\epsilon}} \\ &\leq C w^{1+\epsilon}(S_\beta) \left(\frac{A_\gamma(E)}{A_\gamma(S_\beta)} \right)^{\frac{\epsilon}{1+\epsilon}} \end{aligned}$$

□

We may now complete the proof of Theorem 5.1. If $\beta \in \mathcal{D}$ with $\beta = (n, k)$, then define \widetilde{S}_β to be

$$\widetilde{S}_\beta = S_{(n, k-1)} \cup S_{(n, k)} \cup S_{(n, k+1)}.$$

Fix $u \in \mathbb{D}$ and pick $\beta \in \mathcal{D}$ such that $u \in T_\beta$. Because of Lemma 5.8, we may assume that $d(\beta) \geq 2$. For any $o < \eta \leq \beta$, pick $\tilde{\eta} \leq \eta \leq \beta$ where $d(\eta) = d(\tilde{\eta}) + 1$. Then by (5.4) and the definition of \widetilde{S}_η , we have that

$$\sup_{z \in \mathbb{D} \setminus \widetilde{S}_\eta} |k_u(z)|^2 \lesssim 2^{-d(\beta)(2+\gamma)} 2^{2d(\eta)(2+\gamma)} \lesssim \frac{1}{A_\gamma(\widetilde{S}_\eta)} 2^{-(d(\beta)-d(\eta))(2+\gamma)}. \quad (5.11)$$

Using (5.11) and the fact that

$$\mathbb{D} = \left(\bigcup_{o < \eta < \beta} \widetilde{S}_\eta \setminus \widetilde{S}_\eta \right) \cup \widetilde{S}_\beta,$$

we have that

$$\begin{aligned} \{B_\gamma(|fk_u^{1-\frac{2}{p}}|^{p+\epsilon_1})(u)\}^{\frac{1}{p+\epsilon_1}} &= \left(\int_{\mathbb{D}} |fk_u^{1-\frac{2}{p}}|^{p+\epsilon_1} |k_u|^2 dA_\gamma \right)^{\frac{1}{p+\epsilon_1}} \\ &\leq \sum_{o < \eta \leq \beta} 2^{-d(\beta)(\frac{2+\gamma}{2})(1-\frac{2}{p})} 2^{d(\eta)(2+\gamma)(1-\frac{2}{p})} \\ &\quad \times 2^{-\frac{2+\gamma}{p+\epsilon_1}(d(\beta)-d(\eta))} \left(\frac{|f|^{p+\epsilon_1}(\widetilde{S_\eta})}{A_\gamma(\widetilde{S_\eta})} \right)^{\frac{1}{p+\epsilon_1}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \{B_\gamma(|f^{-1}k_u^{1-\frac{2}{q}}|^{q+\epsilon_2})(u)\}^{\frac{1}{q+\epsilon_2}} &= \left(\int_{\mathbb{D}} |f^{-1}k_u^{1-\frac{2}{q}}|^{q+\epsilon_2} |k_u|^2 dA_\gamma \right)^{\frac{1}{q+\epsilon_2}} \\ &\lesssim \sum_{o < \eta' \leq \beta} 2^{-d(\beta)(\frac{2+\gamma}{2})(1-\frac{2}{q})} 2^{d(\eta')(2+\gamma)(1-\frac{2}{q})} \\ &\quad \times 2^{-\frac{2+\gamma}{q+\epsilon_2}(d(\beta)-d(\eta'))} \left(\frac{|f|^{-q-\epsilon_2}(\widetilde{S_{\eta'}})}{A_\gamma(\widetilde{S_{\eta'}})} \right)^{\frac{1}{q+\epsilon_2}}. \end{aligned}$$

Combining these two inequalities gives us that

$$\begin{aligned} \{B_\gamma(|fk_u^{1-\frac{2}{p}}|^{p+\epsilon_1})(u)\}^{\frac{1}{p+\epsilon_1}} \{B_\gamma(|f^{-1}k_u^{1-\frac{2}{q}}|^{q+\epsilon_2})(u)\}^{\frac{1}{q+\epsilon_2}} \\ \lesssim \sum_{o < \eta, \eta' \leq \beta} 2^{d(\eta)(2+\gamma)(1-\frac{2}{p})} 2^{d(\eta')(2+\gamma)(1-\frac{2}{q})} 2^{-\frac{2+\gamma}{p+\epsilon_1}(d(\beta)-d(\eta))} \\ \times 2^{-\frac{2+\gamma}{q+\epsilon_2}(d(\beta)-d(\eta'))} \left(\frac{|f|^{p+\epsilon_1}(\widetilde{S_\eta})}{A_\gamma(\widetilde{S_\eta})} \right)^{\frac{1}{p+\epsilon_1}} \left(\frac{|f|^{-q-\epsilon_2}(\widetilde{S_{\eta'}})}{A_\gamma(\widetilde{S_{\eta'}})} \right)^{\frac{1}{q+\epsilon_2}} \end{aligned}$$

Now observe that if $\eta, \eta' \leq \beta$, then we either have that $\eta \leq \eta'$ or $\eta' \leq \eta$. Thus, without loss of generality, we need to bound the following quantity by a constant that is independent of $\beta \in \mathcal{D}$:

$$\begin{aligned} \sum_{o < \eta \leq \eta' \leq \beta} 2^{d(\eta)(2+\gamma)(1-\frac{2}{p})} 2^{d(\eta')(2+\gamma)(1-\frac{2}{q})} 2^{-\frac{2+\gamma}{p+\epsilon_1}(d(\beta)-d(\eta))} \\ \times 2^{-\frac{2+\gamma}{q+\epsilon_2}(d(\beta)-d(\eta'))} \left(\frac{|f|^{p+\epsilon_1}(\widetilde{S_\eta})}{A_\gamma(\widetilde{S_\eta})} \right)^{\frac{1}{p+\epsilon_1}} \left(\frac{|f|^{-q-\epsilon_2}(\widetilde{S_{\eta'}})}{A_\gamma(\widetilde{S_{\eta'}})} \right)^{\frac{1}{q+\epsilon_2}} \end{aligned} \quad (5.12)$$

and we need to do the same when the above sum is taken over $\{\eta, \eta' \in \mathcal{D} : o < \eta' \leq \eta \leq \beta\}$.

We first estimate (5.12) for $\eta \leq \eta' \leq \beta$. Note that that

$$\frac{1}{A_\gamma(\widetilde{S_{\eta'}})} \approx 2^{(d(\eta')-d(\eta))(2+\gamma)} \frac{1}{A_\gamma(\widetilde{S_\eta})}. \quad (5.13)$$

Moreover, since the conclusion of Lemma 5.5 holds when $\widetilde{S_\eta}$ replaces S_η for any $\eta \in \mathcal{D}$, it is not difficult to check that the conclusion of Lemma 5.9 holds when $w = |f|^{-q}$ and $\widetilde{S_\eta}$ replaces S_η . Thus, since $\widetilde{S_{\eta'}} \subseteq \widetilde{S_\eta}$, we have that

$$\int_{\widetilde{S_{\eta'}}} |f|^{-q-\epsilon_2} dA_\gamma \lesssim 2^{-(d(\eta')-d(\eta))(2+\gamma)\left(\frac{\epsilon_2}{q+\epsilon_2}\right)} \int_{\widetilde{S_\eta}} |f|^{-q-\epsilon_2} dA_\gamma \quad (5.14)$$

for small enough ϵ_2 . Also, an application of Lemma 5.5 and Lemma 5.8 (where again $\widetilde{S_\eta}$ replaces S_η) gives us that

$$\left(\frac{|f|^{p+\epsilon_1}(\widetilde{S_\eta})}{A_\gamma(\widetilde{S_\eta})} \right)^{\frac{1}{p+\epsilon_1}} \left(\frac{|f|^{-q-\epsilon_2}(\widetilde{S_\eta})}{A_\gamma(\widetilde{S_\eta})} \right)^{\frac{1}{q+\epsilon_2}} \leq C \quad (5.15)$$

where C is independent of $\eta \in \mathcal{D}$.

Plugging (5.13), (5.14), and (5.15) into (5.12) gives us that

$$\begin{aligned} & \sum_{0 < \eta \leq \eta' \leq \beta} 2^{d(\eta)(2+\gamma)\left(1-\frac{2}{p}\right)} 2^{d(\eta')(2+\gamma)\left(1-\frac{2}{q}\right)} 2^{-\frac{2+\gamma}{p+\epsilon_1}(d(\beta)-d(\eta))} \\ & \quad \times 2^{-\frac{2+\gamma}{q+\epsilon_2}(d(\beta)-d(\eta'))} \left(\frac{|f|^{p+\epsilon_1}(\widetilde{S_\eta})}{A_\gamma(\widetilde{S_\eta})} \right)^{\frac{1}{p+\epsilon_1}} \left(\frac{|f|^{-q-\epsilon_2}(\widetilde{S_{\eta'}})}{A_\gamma(\widetilde{S_{\eta'}})} \right)^{\frac{1}{q+\epsilon_2}} \\ & \lesssim \sum_{\eta \leq \eta' \leq \beta} 2^{d(\eta)(2+\gamma)\left(1-\frac{2}{p}\right)} 2^{d(\eta')(2+\gamma)\left(1-\frac{2}{q}\right)} 2^{\frac{2+\gamma}{q+\epsilon_2}(d(\eta')-d(\eta))} \\ & \quad \times 2^{-\frac{\epsilon_2}{(q+\epsilon_2)^2}(d(\eta')-d(\eta))(2+\gamma)} 2^{-\frac{2+\gamma}{p+\epsilon_1}(d(\beta)-d(\eta))} 2^{-\frac{2+\gamma}{q+\epsilon_2}(d(\beta)-d(\eta'))} \\ & = \sum_{\eta \leq \eta' \leq \beta} 2^{-(d(\beta)-d(\eta'))(2+\gamma)} 2^{-(d(\eta')-d(\eta))\left(\frac{\epsilon_2(2+\gamma)}{(q+\epsilon_2)^2} - \frac{2+\gamma}{q+\epsilon_2} + \frac{2+\gamma}{q}\right)} \\ & \quad \times 2^{-(d(\beta)-d(\eta))\left(\frac{2+\gamma}{p+\epsilon_1} - \frac{2+\gamma}{p}\right)} 2^{-(d(\beta)-d(\eta'))\left(\frac{2+\gamma}{q+\epsilon_2} - \frac{2+\gamma}{q}\right)} \\ & = \sum_{\eta' \leq \beta} 2^{-(d(\beta)-d(\eta'))(2+\gamma)\left(\frac{1}{p+\epsilon_1} + \frac{1}{q+\epsilon_2}\right)} \\ & \quad \times \sum_{\eta \leq \eta'} 2^{-(d(\eta')-d(\eta))(2+\gamma)\left(\frac{\epsilon_2}{(q+\epsilon_2)^2} - \frac{\epsilon_1}{p(p+\epsilon_1)} + \frac{\epsilon_2}{q(q+\epsilon_2)}\right)} \end{aligned} \quad (5.16)$$

Similarly, we have that

$$\begin{aligned}
 & \sum_{0 < \eta' \leq \eta \leq \beta} 2^{d(\eta)(2+\gamma)\left(1-\frac{2}{p}\right)} 2^{d(\eta')(2+\gamma)\left(1-\frac{2}{q}\right)} 2^{-\frac{2+\gamma}{p+\epsilon_1}(d(\beta)-d(\eta))} \\
 & \quad \times 2^{-\frac{2+\gamma}{q+\epsilon_2}(d(\beta)-d(\eta'))} \left(\frac{|f|^{p+\epsilon_1}(\widetilde{S_{\tilde{\eta}}})}{A_{\gamma}(\widetilde{S_{\tilde{\eta}}})} \right)^{\frac{1}{p+\epsilon_1}} \left(\frac{|f|^{-q-\epsilon_2}(\widetilde{S_{\tilde{\eta}'})})}{A_{\gamma}(\widetilde{S_{\tilde{\eta}'})})} \right)^{\frac{1}{q+\epsilon_2}} \\
 & \lesssim \sum_{\eta \leq \beta} 2^{-(d(\beta)-d(\eta))(2+\gamma)\left(\frac{1}{p+\epsilon_1}+\frac{1}{q+\epsilon_2}\right)} \\
 & \quad \times \sum_{\eta' \leq \eta} 2^{-(d(\eta)-d(\eta'))(2+\gamma)\left(\frac{\epsilon_1}{(p+\epsilon_1)^2}-\frac{\epsilon_2}{q(q+\epsilon_2)}+\frac{\epsilon_1}{p(p+\epsilon_1)}\right)} \quad (5.17)
 \end{aligned}$$

Clearly the sums (5.16) and (5.17) converge to a sum that has an upper bound independent of $\beta \in \mathcal{D}$ if we simultaneously have

$$\begin{cases} \frac{\epsilon_2}{(q+\epsilon_2)^2} + \frac{\epsilon_2}{q(q+\epsilon_2)} > \frac{\epsilon_1}{p(p+\epsilon_1)} \\ \frac{\epsilon_1}{(p+\epsilon_1)^2} + \frac{\epsilon_1}{p(p+\epsilon_1)} > \frac{\epsilon_2}{q(q+\epsilon_2)} \end{cases}.$$

Moreover, both of these are trivially satisfied if $\frac{\epsilon_2}{q(q+\epsilon_2)} = \frac{\epsilon_1}{p(p+\epsilon_1)}$ or $\epsilon_1 = \frac{\epsilon_2 p^2}{q^2 + \epsilon_2 q - \epsilon_2 p}$ and so the proof is complete so long as $\epsilon_2 > 0$ is set small enough.

Finally in this paper, we will prove Propositions 4.1 and 4.4, starting with Proposition 4.4. The proof is similar to the proof of Theorem 3.2.2 in [10], though we include it since some of the details are different. Let $d(u, v)$ denote the non-isotropic metric on $\partial\mathbb{B}_n$ given by $d(u, v) = |1 - u \cdot v|^{\frac{1}{2}}$ and let $B = B(u, r)$ denote a ball in this metric. It is well known (see [26]) that $B(u, r) = \partial\mathbb{B}_n$ when $r \geq \sqrt{2}$ and that there exists $C > 0$ independent of r and u such that

$$\frac{1}{C} r^{2n} \leq \sigma(B(u, r)) \leq C r^{2n} \quad (5.18)$$

where σ is the canonical surface measure on $\partial\mathbb{B}_n$.

Pick some large $M > 0$ such that $C^2 M^{-n} \leq \frac{1}{2}$ where C is the constant in (5.18). For some fixed $1/M < |z| < 1$, pick $J \in \mathbb{N}$ such that $M^{-J-1} \leq 1 - |z| < M^{-J}$, and let $B_k = B(z/|z|, M^{\frac{k-J}{2}})$ for $k \in \{0, 1, \dots, J+1\}$. Now, for any $0 \leq t \leq 1$, $0 \leq a \leq 1$, and $\theta \in \mathbb{R}$, we have

$$|1 - tae^{i\theta}|^2 = t|1 - ae^{i\theta}|^2 + (1-t)(1 - ta^2) \geq t|1 - ae^{i\theta}|^2.$$

Thus, if $\zeta \in \partial\mathbb{B}_n \setminus B_k$, then writing $\zeta \cdot z = tae^{i\theta}$ where $t = |z|$ and $ae^{i\theta} = \zeta \cdot (z/|z|)$ gives us that

$$|k_z(\zeta)| = \frac{(1 - |z|^2)^{n/2}}{|1 - \zeta \cdot z|^n} \lesssim M^{-\frac{nJ}{2}} M^{-n(k-J)} \lesssim \frac{M^{-\frac{nk}{2}}}{(\sigma(B_{k+1}))^{\frac{1}{2}}}$$

Also, if $\zeta \in B_0$, then we have that $|k_z(\zeta)| \approx (\sigma(B_0))^{-\frac{1}{2}}$

Thus, we have that

$$\begin{aligned} & \left(\int_{\partial\mathbb{B}_n} w |k_z|^p d\sigma \right) \left(\int_{\partial\mathbb{B}_n} w^{-\frac{1}{p-1}} |k_z|^q d\sigma \right)^{p-1} \\ &= \sum_{k,k'=0}^J \left(\int_{B_{k+1} \setminus B_k} w |k_z|^p d\sigma \right) \left(\int_{B_{k'+1} \setminus B_{k'}} w^{-\frac{1}{p-1}} |k_z|^q d\sigma \right)^{p-1} \\ &+ \left(\int_{B_0} w |k_z|^p d\sigma \right) \left(\int_{B_0} w^{-\frac{1}{p-1}} |k_z|^q d\sigma \right)^{p-1} \\ &\lesssim \sum_{k,k'=-1}^J \frac{M^{-\frac{npk}{2}}}{(\sigma(B_{k+1}))^{\frac{p}{2}}} \frac{M^{-\frac{npk'}{2}}}{(\sigma(B_{k'+1}))^{\frac{p}{2}}} \left(\int_{B_{k+1}} w d\sigma \right) \left(\int_{B_{k'+1}} w^{-\frac{1}{p-1}} d\sigma \right)^{p-1} \end{aligned} \quad (5.19)$$

Now break the sum in (5.19) into two sums, the first of which is taken over $k \leq k'$ and the second over $k' < k$. In the first case, we have that

$$\frac{M^{-\frac{npk}{2}}}{(\sigma(B_{k+1}))^{\frac{p}{2}}} \lesssim \frac{M^{-\frac{npk}{2}} M^{\frac{np(k'-k)}{2}}}{(\sigma(B_{k'+1}))^{\frac{p}{2}}} = \frac{M^{-npk} M^{\frac{npk'}{2}}}{(\sigma(B_{k'+1}))^{\frac{p}{2}}}.$$

Moreover, similar to Lemma 5.6, we have that $\frac{w(B_{k+1})}{w(B_{k'+1})} \leq \delta_1^{k'-k}$ where $\delta_1 = 1 - (2^p \|w\|_{A_p})^{-1}$. Thus, we have that

$$\begin{aligned} & \sum_{k \leq k'}^J \frac{M^{-\frac{npk}{2}}}{(\sigma(B_{k+1}))^{\frac{p}{2}}} \frac{M^{-\frac{npk'}{2}}}{(\sigma(B_{k'+1}))^{\frac{p}{2}}} \left(\int_{B_{k+1}} w d\sigma \right) \left(\int_{B_{k'+1}} w^{-\frac{1}{p-1}} d\sigma \right)^{p-1} \\ &\lesssim \sum_{k \leq k'}^J \frac{M^{-npk} \delta_1^{k'-k}}{(\sigma(B_{k'+1}))^p} \left(\int_{B_{k'+1}} w d\sigma \right) \left(\int_{B_{k'+1}} w^{-\frac{1}{p-1}} d\sigma \right)^{p-1} \\ &= \sum_{k=-1}^{J+1} M^{-npk} \sum_{k'=k}^J \delta_1^{k'-k} \left(\frac{1}{(\sigma(B_{k'+1}))} \int_{B_{k'+1}} w d\sigma \right) \left(\frac{1}{(\sigma(B_{k'+1}))} \int_{B_{k'+1}} w^{-\frac{1}{p-1}} d\sigma \right)^{p-1} \\ &\lesssim \|w\|_{A_p}^2 \end{aligned}$$

Similarly, for $k' < k$ we have that $\frac{w^{-\frac{1}{p-1}}(B_{k'+1})}{w^{-\frac{1}{p-1}}(B_{k+1})} \leq \delta_2^{k-k'}$ where $\delta_2 = 1 - (2^q \|w\|_{A_p}^{\frac{q}{p}})^{-1}$, so that

$$\begin{aligned} \sum_{k' < k}^J \frac{M^{-\frac{npk}{2}}}{(\sigma(B_{k+1}))^{\frac{p}{2}}} \frac{M^{-\frac{npk'}{2}}}{(\sigma(B_{k'+1}))^{\frac{p}{2}}} \left(\int_{B_{k+1}} w \, d\sigma \right) \left(\int_{B_{k'+1}} w^{-\frac{1}{p-1}} \, d\sigma \right)^{p-1} \\ \lesssim \frac{\|w\|_{A_p}}{1 - \delta_2^{p-1}} \lesssim \|w\|_{A_p}^{1+\frac{q}{p}} \end{aligned}$$

which proves Proposition 4.4.

Now to prove proposition 4.1, let d be the pseudo-metric $d(z, u) = ||z| - |u|| + |1 - \frac{z}{|z|} \cdot \frac{u}{|u|}|$ on \mathbb{B}_n . According to Lemma 2 in [3], there exists $C > 0$ such that

$$\frac{1}{C} r^{n+1+\gamma} \leq v_\gamma(B(u, r)) \leq C r^{n+1+\gamma}$$

whenever $r \geq 1 - |u|$ (and $u \in \mathbb{B}_n$.) As before, pick some large $M > 0$ where $C^2 M^{-(n+1+\gamma)} \leq \frac{1}{2}$ and for some fixed $\frac{1}{M} < |z| < 1$, pick J where $M^{-J-1} \leq 1 - |z| < M^{-J}$ and let $B_k = B(z, M^{k-J})$ for $k \in \{0, 1, \dots, J+1\}$. Note that we clearly have $z/|z| \in B_k$ for each k and note that $M^{k-J} \geq 1 - |z|$. The proof of Proposition 4.1 is now almost identical to the proof of Proposition 4.4 above.

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